POINTED HOPF ALGEBRAS AS COCYCLE DEFORMATIONS

L. GRUNENFELDER AND M. MASTNAK

Abstract. We show that all finite dimensional pointed Hopf algebras with the same diagram in the classification scheme of Andruskiewitsch and Schneider are cocycle deformations of each other. This is done by giving first a suitable characterization of such Hopf algebras, which allows for the application of results by Masuoka about Morita-Takeuchi equivalence and by Schauenburg about Hopf Galois extensions. We also outline a method to describe the deforming cocycles involved using the exponential map and its q-analogue.

0. Introduction

Finite dimensional pointed Hopf algebras over an algebraically closed field of characteristic zero, particularly when the group of points is abelian, have been studied quite extensively with various methods in [AS, BDG, Gr, Mu]. The most far reaching results as yet in this area have been obtained in [AS], where a large class of such Hopf algebras are classified. In the present paper we show, among other things, that all Hopf algebras in this class can be obtained by cocycle deformations from Radford biproducts of the form $B(V)\#kG$, where $B(V)$ is the Nichols algebra of the Yetter-Drinfeld kG-module V.

After Kaplansky's tenth conjecture concerning the finiteness of the set of isomorphism classes of Hopf algebras of a given finite dimension had been refuted, a weakened version of that conjecture has been proposed [Ma, Di], namely that there are only finitely many quasi-isomorphism classes of Hopf algebras of a given (odd) dimension. Two Hopf algebras are said to be quasi-isomorphic if they have equivalent comodule categories. In even dimensions the weakened conjecture has been disproved in [EG, Gra], where it is shown that there are infinitely many isomorphism classes of Hopf algebras of dimension 32. Our results confirm the conjecture for a large class of pointed Hopf algebras of odd dimension. If H is a Hopf algebra with coradical a Hopf subalgebra then the graded coalgebra $\operatorname{gr}^c H$ associated with the coradical filtration is a graded Hopf algebra and its elements of positive degree form the radical. If the radical of H is a Hopf ideal then the graded algebra associated with the radical filtration is a graded Hopf algebra with $\operatorname{Cor}(\operatorname{gr}^r H) \cong H/\operatorname{Rad} H$. In either case we have $\operatorname{gr} H \cong R\#H_0$, where H_0 is the

Research supported in part by NSERC.

degree zero part and R is the braided Hopf algebra of coinvariants or invariants, respectively.

The Nichols algebra $B(V)$ of a crossed kG-module V is a connected graded braided Hopf algebra. The Radford biproduct $H(V) = B(V)\#kG$ is an ordinary graded Hopf algebra with coradical kG and the elements of positive degree form a Hopf ideal (the graded radical). A lifting of $H(V)$ is a pointed Hopf algebra H for which $\operatorname{gr}^c H \cong H(V)$. Such liftings are obtained by deforming the multiplication of $H(V)$. The lifting problem for V asks for the classification of all liftings of $H(V)$. This problem, together with the characterization of $B(V)$ and $H(V)$, have been solved by Andruskiewitsch and Schneider in [AS] for a large class of crossed kG-modules of finite Cartan type. It allows them to classify all finite dimensional pointed Hopf algebras A for which the order of the abelian group of points has no prime factors < 11. In this paper we find a description of these lifted Hopf algebras, which is suitable for the application of a result of Masuoka about Morita-Takeuchi equivalence [Ma] and of Schauenburg about Hopf Galois extensions [Sch], to prove that all liftings of a given $H(V)$ in this class are cocycle deformations of each other. As a result we see here that in the class of finite dimensional pointed Hopf algebras classified by Andruskiewitsch an Schneider [AS] all Hopf algebras H with isomorphic associated graded Hopf algebra $\operatorname{gr}^c H$ are monoidally Morita-Takeuchi equivalent, and therefore cocycle deformations of each other. For some special cases such results have been obtained in [Ma, Di, BDR].

In Section 1 we give a short review of braided spaces, braided Hopf algebras, Nichols algebras and bosonization (among other things, we prove Theorem 1.3 in which we give a new way of looking at the quantum symmetrizer). Most of this is done in preparation for a useful characterization of the liftings for a large class of crossed modules over finite abelian groups in Section 2. With this characterization it is then possible in Section 3 to prove that liftings are Morita-Takeuchi equivalent (Theorem 3.3) by using Masuoka's pushout construction, and that they are cocycle deformations of each other (Corollary 3.4) by a result of Schauenburg. Cocycle deformations as well as their relation to Hochschild cohomology are discussed in Section 4. See [GM1] and [Gr2] for a more detailed account of the cohomological aspects. In particular, a complete description is given of all relevant cocycles for the quantum linear case, where exponential map and its q-analogue play a prominent role. In Subsection 4.6 we outline the main ideas of [GM2], where we, among other things, use the exponential map, its q-analogue, and the theory of Singer extensions, to explicitly describe the deforming cocycles for all Anruskiewitsch-Schneider Hopf algebras. Explicit examples of deforming cocycles are presented in Section 5. We recommend to the reader to start by looking at Example 5.1 where the theory is illustrated in great detail on the smallest nontrivial example.

Acknowledgements. We thank A. Masuoka for correcting an error after reading an early version [GM1] of a part of this work. The second author is very grateful for useful conversations with M. Beattie.

1. Braided Hopf algebras and the (bi-)crossproduct

Our main interest in this paper are pointed Hopf algebras. We also briefly look at the dual notion, that is copointed Hopf algebras. Here we review some relevant prerequisites and facts.

1.1. **Pointed and copointed Hopf algebras.** A Hopf algebra H is pointed if its coradical $\operatorname{Cor} H$ is equal to the group algebra of the group of points $G(H)$. In this case the coradical filtration is an ascending Hopf algebra filtration and the associated graded Hopf algebra $\operatorname{gr}^c H$ has the obvious injection $\kappa^c\colon kG \to \operatorname{gr}^c H$ and projection $\pi^c\colon \operatorname{gr}^c H \to kG$ such that $\pi^c\kappa^c = 1$.

We say that H is copointed if its radical $\operatorname{Rad} H$ is a Hopf ideal and $H/\operatorname{Rad} H$ is a group algebra kG. Here the radical filtration is an descending Hopf algebra filtration and again the associated graded Hopf algebra $\operatorname{gr}^r H$ has the obvious projection $\pi^r\colon \operatorname{gr}^r H \to kG$ and an injection $\kappa^r\colon kG \to \operatorname{gr}^r H$ such that $\pi^r\kappa^r = 1$.

In both cases above $\operatorname{gr} H$ is graded, pointed and copointed, and by [Ra] $\operatorname{gr} H \cong A\#kG$, where $A = \{\, x \in H \mid (\pi \otimes 1)\Delta(x) = 1 \otimes x\}$ is the graded connected braided Hopf algebra of coinvariants.

Lemma 1.1. *If the Hopf algebra H is pointed and copointed then* $\operatorname{Cor} H \cong H/\operatorname{Rad} H$ *and H is a Hopf algebra with a projection. Moreover, $R\#kG \cong H$, where $R = \{\, x \in \operatorname{gr} H \mid (p \otimes 1)\Delta(x) = 1 \otimes x\}$ is the connected braided Hopf algebra of coinvariants of H. This is the case in particular for* $\operatorname{gr}^c H$ *and for* $\operatorname{gr}^r H$ *when H is pointed or copointed, respectively.*

Proof. A surjective coalgebra map $\eta\colon C \to D$, where $D = \operatorname{Cor}(D)$, maps $\operatorname{Cor} C$ onto $\operatorname{Cor} D$ [Mo]. Thus, the composite $\operatorname{Cor} H \to H \to H/\operatorname{Rad} H$ is a bijection. The isomorphism is that of [Ra]. □

1.2. **Braidings.** A braided monoidal category $\mathcal{V}$ is a monoidal category together with a natural morphism $c\colon V \otimes W \to W \otimes V$ such that

(1) $c_{k,V} = \tau = c_{V,k}$,
(2) $c_{U\otimes V,W} = (c_{U,W} \otimes 1)(1 \otimes c_{V,W})$,
(3) $c_{U,V\otimes W} = (1 \otimes c_{U,W})(c_{U,V} \otimes 1)$,
(4) $c(f \otimes g) = (g \otimes f)c$.

Braided algebras, braided coalgebras and braided Hopf algebras are now defined with this tensor product and braiding in mind. The compatibility condition $\Delta \mathrm{m} = (m\otimes m)(1\otimes c\otimes 1)(\Delta\otimes\Delta)$ between multiplication and comultiplication in a braided Hopf algebra A involves the braiding $c\colon A \otimes A \to A \otimes A$, so that the diagram

$$\begin{tikzcd} A\otimes A \arrow[r, "(1\otimes c\otimes 1)(\Delta\otimes\Delta)"] \arrow[d, "m"'] & A\otimes A\otimes A\otimes A \arrow[d, "m\otimes m"'] \\ A \arrow[r, "\Delta"] & A\otimes A \end{tikzcd}$$

commutes, i.e. multiplication and unit are morphisms of braided coalgebras or, equivalently, comultiplication and counit are maps of braided algebras.

In the present paper $\mathcal{V}$ is a category of braided vector spaces, where the braidings $c\colon V \otimes V \to V \otimes V$ are of finite abelian group type, so that $c\colon V \otimes V \to V \otimes V$ is of diagonal type.

1.3. **Primitives and indecomposables.** The vector space of primitives

$$P(A) = \{ y \in A |\, \Delta(y) = y \otimes 1 + 1 \otimes y\} \cong \ker(\tilde{\Delta})\colon A/k \to A/k \otimes A/k$$

of a braided Hopf algebra A is a braided vector space, since Δ is a map in $\mathcal{V}$. The c-bracket map $[-,-]_c = m(1 \otimes 1 - c)\colon A \otimes A \to A$ restricted to $P(A)$ satisfies $\Delta[x,y]_c = [x,y]_c \otimes 1 + (1-c^2)x \otimes y + 1 \otimes [x,y]_c$; in particular $[x,y]_c \in P(A)$ if and only $c^2(x \otimes y) = x \otimes y$. Moreover, if $x \in P(A)$ and $c(x \otimes x) = qx \otimes x$ then $\Delta x^n = \sum_{i+j=n} \binom{n}{i}_q x^i \otimes x^j$, where $\binom{n}{i}_q = \frac{n_q!}{i_q!(n-i)_q!}$ are the q-binomial coefficients (the Gauss polynomials) for q, $m_q! = 1_q 2_q ... m_q$ with $j_q = 1 + q + ... + q^{j-1}$ if $j > 0$ and $0_q! = 1$. If $q = 1$ then $j_q = j$ and we have the ordinary binomial coefficients, otherwise $j_q = \frac{1-q^j}{1-q}$. In particular, if q has order n then $\binom{n}{i}_q = 0$ for $0 < i < n$, and hence $x^n \in P(A)$.

Lemma 1.2. *Let $\{x_i\}$ be a basis of $P(A)$ such that $c(x_i \otimes x_j) = q_{ji} x_j \otimes x_i$. If $q_{ji} q_{ij} q_{ii}^{r-1} = 1$ then $\operatorname{ad} x_i^r(x_j)$ is primitive.*

Proof. See for example [AS1, Appendix 1]. □

The vector space of indecomposables

$$Q(A) = JA/JA^2 = \operatorname{cok}(\tilde{m}\colon JA \otimes JA \to JA),$$

where $JA = \ker(\epsilon)$, is a braided vector space as well. The c-cobracket map $\delta_c = (1 \otimes 1 - c)\Delta\colon A \to A \otimes A$ restricts to JA, since

$$\Delta(\bar{a}) = \bar{a} \otimes 1 + 1 \otimes \bar{a} + \sum \bar{a}_i \otimes \bar{b}_i$$

and hence

$$\delta(\bar{a}) = \sum (\bar{a}_i \otimes \bar{b}_i - c(\bar{a}_i \otimes \bar{b}_i))$$

is in $JA \otimes JA$ for every $\bar{a} = a - \epsilon(a) \in JA$. Moreover,

$$\delta(\bar{a}\bar{b}) - (\bar{a} \otimes \bar{b} - c^2(\bar{a} \otimes \bar{b}))$$

is in $JA^2 \otimes JA + JA \otimes JA^2$. In particular, if $c^2(\bar{a} \otimes \bar{b}) = \bar{a} \otimes \bar{b}$, then $\delta(\bar{a} \otimes \bar{b}) \in JA^2 \otimes JA + JA \otimes JA^2$.

1.4. **The free and the cofree graded braided Hopf algebras.** The forgetful functor $U\colon \mathrm{Alg}_c \to \mathcal{V}_c$ has a left-adjoint $\mathcal{A}\colon \mathcal{V}_c \to \mathrm{Alg}_c$ and the forgetful functor $U\colon \mathrm{Coalg}_c \to \mathcal{V}_c$ has a right-adjoint $\mathcal{C}\colon \mathcal{V}_c \to \mathrm{Coalg}_c$, the free braided graded algebra functor and the cofree graded braided coalgebra functor, respectively. Moreover, there is a natural transformation $\mathcal{S}\colon \mathcal{A} \to \mathcal{C}$, the shuffle map or quantum symmetrizer. They can be described as follows.

If (V, μ, δ) is a braided vector space then the tensor powers $T_0(V) = k$, $T_{n+1}(V) = V \otimes T_n(V)$ are braided vector spaces as well and so is $T(V) = \oplus_n T_n(V)$. The ordinary tensor algebra structure makes $T(V)$ the free connected graded braided algebra, and the ordinary tensor coalgebra structure makes it the cofree connected graded braided coalgebra.

By the universal property of the graded braided tensor algebra $T(V)$ the linear map $\Delta_1 = \mathrm{incl}\,\mathrm{diag}\colon V \to T(V) \otimes T(V)$, $\Delta_1(v) = v \otimes 1 + 1 \otimes v$, induces the c-shuffle comultiplication $\Delta_{\mathcal{A}}\colon T(V) \to T(V) \otimes T(V)$, which is a homomorphism of braided algebras, so that $\Delta_{\mathcal{A}} m = (m \otimes m)(1 \otimes c \otimes 1)(\Delta_{\mathcal{A}} \otimes \Delta_{\mathcal{A}})$. Moreover, the linear map $s_1\colon V \to T(V)$, $s_1(v) = -v$, extends uniquely to a c-antipode $s_{\mathcal{A}}\colon T(V) \to T(V)$, such that $s_{\mathcal{A}} m = m(s_{\mathcal{A}} \otimes s_{\mathcal{A}})c$, $\Delta_{\mathcal{A}} s_{\mathcal{A}} = c(s_{\mathcal{A}} \otimes s_{\mathcal{A}})\Delta_{\mathcal{A}}$ and $m(1 \otimes s_{\mathcal{A}})\Delta_{\mathcal{A}} = \iota\epsilon = m(s_{\mathcal{A}} \otimes 1)\Delta_{\mathcal{A}}$, thus making $\mathcal{A}(V) = (T(V), m, \Delta_{\mathcal{A}}, s_{\mathcal{A}})$ the free connected graded braided Hopf algebra. This defines a functor $\mathcal{A}\colon \mathcal{V}_c \to Hopf_c$, left-adjoint to the space of primitives functor $P\colon Hopf_c \to \mathcal{V}_c$.

On the other hand, by the universal property of the cofree connected graded braided coalgebra $T(V)$ there is a unique c-shuffle multiplication $m_{\mathcal{C}}\colon T(V) \otimes T(V) \to T(V)$, which is the homomorphism of braided coalgebras induced by the linear map $m_1 = +\,\mathrm{proj}\colon T(V) \otimes T(V) \to V$, so that $\Delta m_{\mathcal{C}} = (m_{\mathcal{C}} \otimes m_{\mathcal{C}})(1 \otimes c \otimes 1)(\Delta \otimes \Delta)$. The linear map $s_1 = -\,\mathrm{proj}\colon T(V) \to V$ induces uniquely a c-antipode $s_{\mathcal{C}}\colon T(V) \to T(V)$, such that $s_{\mathcal{C}} m_{\mathcal{C}} = m_{\mathcal{C}}(s_{\mathcal{C}} \otimes s_{\mathcal{C}})c$, $\Delta s_{\mathcal{C}} = c(s_{\mathcal{C}} \otimes s_{\mathcal{C}})\Delta$ and $m_{\mathcal{C}}(1 \otimes s_{\mathcal{C}})\Delta = \iota\epsilon = m_{\mathcal{C}}(s_{\mathcal{C}} \otimes 1)\Delta$, making $\mathcal{C}(V) = (T(V), \Delta, m_{\mathcal{C}}, s_{\mathcal{C}})$ the cofree connected graded braided Hopf algebra. The functor $\mathcal{C}\colon \mathcal{V}_c \to Hopf_c$ is right-adjoint to the space of indecomposables functor $Q\colon Hopf_c \to \mathcal{V}_c$.

Theorem 1.3. *There is a natural transformation*

$$\mathcal{S}\colon \mathcal{A} \to \mathcal{C},$$

the quantum symmetrizer, such that

$$\mathcal{B}(V) \cong \mathcal{A}(V)/\ker(\mathcal{S}) \cong \operatorname{im}\mathcal{S} \subset \mathcal{C}(V)$$

is the Nichols algebra of V and $Q\mathcal{B}(V) \cong V \cong P\mathcal{B}(V)$. In particular, the Hopf ideal of $\mathcal{A}(V)$ generated by the primitives of degree ≥ 2 is contained in $\ker\mathcal{S}$.

Proof. The adjunctions just described provide natural isomorphisms

$$\mathcal{V}_c(Q\mathcal{A}(V), W) \cong Hopf_c(\mathcal{A}(V), \mathcal{C}(W)) \cong \mathcal{V}_c(V, P\mathcal{C}(W)).$$

By construction we also have

$$Q\mathcal{A}(V) \cong V \quad , \quad W \cong P\mathcal{C}(W),$$

The resulting natural isomorphism

$$\theta_{V,W}\colon \mathcal{V}_c(V,W) \to Hopf_c(\mathcal{A}(V), \mathcal{C}(W))$$

sends the identity morphism of V to the quantum symmetrizer

$$\mathcal{S} = \theta_{V,V}(1_V)\colon \mathcal{A}(V) \to \mathcal{C}(V).$$

The image of $\mathcal{S}$ is the Nichols algebra

$$\mathcal{B}(V) \cong \mathcal{A}(V)/\ker(\mathcal{S}) \cong \operatorname{im}\mathcal{S} \subset \mathcal{C}(V)$$

and $Q\mathcal{B}(V) \cong V \cong P\mathcal{B}(V)$. Moreover, since $\mathcal{S}$ is graded, it follows that $\mathcal{S}(y) = 0$ for every primitive $y \in \mathcal{A}(V)$ of degree ≥ 2. $\square$

An explicit description of the quantum symmetrizer can be obtained directly in term of the action of the braid groups B_n on the tensor powers $V^{\otimes n}$.

1.5. **Crossed modules.** A prime example of a braided monoidal category is the category of crossed H-modules YD_H^H for a Hopf algebra H. A crossed H-module or a Yetter-Drinfeld H-module, (V, μ, δ) is a vector space V with a H-module structure $\mu\colon H \otimes V \to V$, $\mu(h \otimes v) = hv$, and a H-comodule structure $\delta\colon V \to H \otimes V$, $\delta(v) = v_{-1} \otimes v_0$, such that $h\delta(v) = h_1 v_{-1} \otimes h_2 v_0 = (h_1 v)_{-1} h_2 \otimes (h_1 v)_0$, or

$$(m \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \delta) = (m \otimes 1)(1 \otimes \tau)(\delta\mu \otimes 1)(1 \otimes \tau)(\Delta \otimes 1),$$

i.e such that the diagram

$$\begin{tikzcd}
H \otimes V \arrow[r, "\Delta\otimes\delta"] \arrow[d, "(1\otimes\tau)(\Delta\otimes 1)"'] & H \otimes H \otimes H \otimes V \arrow[r, "1\otimes\tau\otimes 1"] & H \otimes H \otimes H \otimes V \arrow[d, "m\otimes\mu"] \\
H \otimes V \otimes H \arrow[r, "\delta\mu\otimes 1"] & H \otimes V \otimes H \arrow[r, "(m\otimes 1)(1\otimes\tau)"] & H \otimes V
\end{tikzcd}$$

commutes. This is the case in particular when $\delta(hv) = h_1 v_{-1} s(h_3) \otimes h_2 v_0$, i.e: when the diagram

$$\begin{tikzcd}
H \otimes V \arrow[r, "(1\otimes\phi\otimes 1)(\Delta\otimes\delta)"] \arrow[d, "\mu"'] & H \otimes H \otimes H \otimes V \arrow[d, "m\otimes\mu"] \\
V \arrow[r, "\delta"] & H \otimes V
\end{tikzcd}$$

commutes, where $\phi = (m \otimes 1)(1 \otimes s \otimes 1)(1 \otimes \tau\Delta)\tau\colon H \otimes H \to H \otimes H$, $\phi(g \otimes h) = hs(g_2) \otimes g_1$. These braided H-modules with the obvious homomorphisms form a braided monoidal category, with the ordinary tensor product of vector spaces together with diagonal action and diagonal coaction. The braiding, given by $c(v \otimes w) = v_{-1}(w) \otimes v_0$,

$$c = (\mu \otimes 1)(1 \otimes \tau)(\delta \otimes 1)\colon V \otimes W \to W \otimes V,$$

clearly satisfies the braiding conditions. The crossed H-module $(k, \mu = \varepsilon \otimes 1, \delta = \iota \otimes 1)$ acts as a unit for the tensor. Moreover, $(H, \mathrm{adj}, \Delta)$ and (H, m, coadj) are crossed H-modules, where $\mathrm{adj}(h \otimes h') = h_1 h' S(h_2)$ and $\mathrm{coadj}(h) = h_1 S(h_3) \otimes h_2$.

If $H = kG$ for some finite abelian group and V is finite dimensional, then the action of G is diagonalizable, so that

$$V \cong \oplus_{g \in G, \chi \in \hat{G}} V_g^\chi,$$

where $V_g^\chi = V_g \cap V^\chi$ with $V_g = \{v \in V | \delta(v) = g \otimes v\}$ and $V^\chi = \{v \in V | gv = \chi(g)v, \forall g \in G\}$.

1.6. **The pushout construction for bi-cross products.** Recall Masuoka's pushout construction for Hopf algebras [Ma], [Gr]. If A is a Hopf algebra then $\mathrm{Alg}(A, k)$ is a group under convolution which acts on A by conjugation as Hopf algebra automorphisms.

Lemma 1.4. *For every Hopf algebra A the group $\mathrm{Alg}(A, k)$ acts on A by 'conjugation' as Hopf algebra automorphisms*

$$\rho\colon \mathrm{Alg}(A, k) \to \mathrm{Aut}_{Hopf}(A)^{op},$$

where $\rho_f = f * 1 * fs$, i.e: $\rho_f(x) = f(x_1)x_2 f(sx_3)$. The image of ρ is a normal subgroup of $\mathrm{Aut}_{Hopf}(A)$.

Proof. It is easy to verify that ρ_f is an Hopf algebra map. The definition of ρ shows that $\rho_{f_1 * f_2} = (f_1 * f_2) * 1 * (f_2 s * f_1 s) = \rho_{f_2}\rho_{f_1}$ and, since $f * fs = \varepsilon = fs * f$, it follows that $\rho_f \rho_{fs} = 1 = \rho_{fs}\rho_f$, so that ρ_f is a Hopf algebra automorphism. If $\phi \in \mathrm{Aut}_{Hopf}(A)$ and $f \in \mathrm{Alg}(A, k)$ then $\phi^{-1}\rho_f \phi = f\phi * 1 * fs\phi = \rho_{f\phi}$, hence the image of ρ is a normal subgroup. □

Two Hopf ideals I and J of A are said to be conjugate if $J = \rho_f(I) = f * I * fs$ for some $f \in \mathrm{Alg}(A, k)$. If $x \in P_{1,g}$ is a $(1, g)$-primitive then

$$\rho_f(x) = f(x) + f(g)x + f(g)gfs(x) = f(g)x + f(x)(g - 1).$$

Theorem 1.5. [Ma, Theorem 2][BDR, Theorem 3.4] *Let A' be a Hopf subalgebra of A. If the Hopf ideals I and J of A' are conjugate and $A/(f * I) \neq 0$ then the quotient Hopf algebras $A/(I)$ and $A/(J)$ by the Hopf ideals in A generated by I and J are monoidally Morita-Takeuchi equivalent, i.e: there exists a k-linear monoidal equivalence between their (left) comodule categories.*

Proof. Masuoka's result [Ma, Theorem 2], that there is a $(A/(I), A/(J))$-biGalois object, namely $A/(f * I)$, holds , provided that $A/(f * I) \neq 0$ ([BDR], Theorem 3.4), and we can invoke [Sch, Corollary 5.7], to see that $A/(I)$ and $A/(J)$ are Morita-Takeuchi equivalent. □

Observe, as Masuoka did [Ma], that the commutative square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B/I & \longrightarrow & A/(I) \end{array}$$

is a pushout of Hopf algebras.

If R is a braided Hopf algebra in the braided category of crossed H-modules then the bi-cross product $R\#H$ is an ordinary Hopf algebra with multiplication

$$(x\#h)(x'\#h') = xh_1(x')\#h_2h'$$

and comultiplication

$$\Delta(x\#h) = x_1\#(x_2)_{-1}h_1 \otimes (x_2)_0\#h_2.$$

The (left) action of H on R induces a (right) action on $\mathrm{Alg}(R,k)$ by $fh(x) = f(hx)$. An algebra map $f\colon R \to k$ is H-invariant if $fh = \varepsilon(h)f$ for all $h \in H$.

Proposition 1.6. *Let K be a Hopf algebra in the braided category of crossed H-modules and let $\mathrm{Alg}_H(K,k)$ be the set of H-invariant algebra maps. Then:*

(1) $\mathrm{Alg}_H(K,k)$ *is a group under convolution.*

(2) *The restriction map* $\mathrm{res}\colon {}_H\mathrm{Alg}_H(K\#H,k) \to \mathrm{Alg}_H(K,k)$, $\mathrm{res}(F) = F\otimes\iota$, *is an isomorphism of groups with inverse given by* $\mathrm{res}^{-1}(f) = f\otimes\varepsilon$.

(3) *The image of the conjugation homomorphism*

$$\Theta = \rho\,\mathrm{res}^{-1}\colon\ \mathrm{Alg}_H(K,k) \to \mathrm{Aut}_{Hopf}(K\#H)^{op}$$

is contained in $\widetilde{\mathrm{Aut}}_{Hopf}(K\#H) = \big\{\, \phi \in \mathrm{Aut}_{Hopf}(K\#H) \,|\, \phi_{|H} = id \big\}$.

Proof. The set of algebra maps $\mathrm{Alg}(K,k)$ may not be a group, but since the coequalizer $K^H = \mathrm{coeq}(\mu, \varepsilon\otimes 1\colon H\otimes K \to K)$ is an ordinary Hopf algebra, $\mathrm{Alg}_H(K,k) \cong \mathrm{Alg}(K^H,k)$ is a group under convolution. More directly, if $f, f' \in \mathrm{Alg}_H(K,k)$ then

$$\begin{aligned} f * f'(xy) &= f\otimes f'(x_1(x_2)_{-1}y_1 \otimes (x_2)_0 y_2) = f(x_1)f'(y_1)f(x_2)f'(y_2) \\ &= (f*f')(x)(f*f')(y) \\ f*f'(hx) &= f\otimes f'(h_1x_1 \otimes h_2x_2) = f(h_1x_1)f'(h_2x_2) = \varepsilon(h)(f*f')(x), \end{aligned}$$

and $f * fs = \varepsilon = fs * f$, so that $\mathrm{Alg}_H(K,k)$ is closed under convolution multiplication and inversion.

For $F \in {}_H\mathrm{Alg}_H(K\#H,k)$ the map $\mathrm{res}(F)\colon K \to k$ is in fact a H-invariant algebra map, since

$$\begin{aligned} \mathrm{res}(F)(hx) &= F(hx\otimes 1) = F((1\otimes h_1)(x\otimes 1)(1\otimes s(h_3)) \\ &= \varepsilon(h)F(x\otimes 1) = \varepsilon(h)\,\mathrm{res}(F)(x) \end{aligned}$$

and

$$\mathrm{res}(F)(xy) = F(xy\otimes 1) = F(x\otimes 1)F(y\otimes 1) = \mathrm{res}(F)(x)\,\mathrm{res}(F)(y).$$

If $F' \in_H \mathrm{Alg}_H(K\#H, k)$ as well, then

$$\begin{aligned} \mathrm{res}(F * F')(x) &= F \otimes F'(x_1 \otimes (x_2)_{-1} \otimes (x_2)_0 \otimes 1) \\ &= F(x_1 \otimes 1)F'(x_2 \otimes 1) = \mathrm{res}(F) * \mathrm{res}(F')(x), \end{aligned}$$

showing that res is a group homomorphism. It is now easy to see that res is invertible and that the inverse is as stated.

As a composite of two group homomorphisms Θ is obviously a group homomorphism. Moreover,

$$\Theta(f)(1 \otimes h) = \mathrm{res}^{-1}(f) * 1 * \mathrm{res}^{-1}(f)s(1 \otimes h) = \varepsilon(h_1)(1 \otimes h_2)\varepsilon(h_3) = 1 \otimes h$$

for $f \in \mathrm{Alg}_H(K, k)$, showing that $\Theta(f)_{|H} = id$. □

Corollary 1.7. *Let R be a braided Hopf algebra in the braided category of crossed H-modules and let K be a braided Hopf subalgebra. If I is a Hopf ideal in K and $f \in \mathrm{Alg}_H(K, k)$ then,*

- *$J = I\#H$ and $J_f = \Theta(f)(J)$ are Hopf ideals in $K\#H$,*
- *$R\#H/(J) = R/(I)\#H$ and $R\#H/(J_f)$ are monoidally Morita-Takeuchi equivalent, if $(R\#H)/(\mathrm{res}^{-1}(f) * J) \neq 0$.*

2. Liftings over finite abelian groups

In this section we give a somewhat different characterization of the class of finite dimensional pointed Hopf algebras classified in [AS], and show that any two such Hopf algebras with isomorphic associated graded Hopf algebras are monoidally Morita-Takeuchi equivalent, and therefore cocycle deformations of each other, as we will point out in the next section.

A datum of finite Cartan type

$$\mathcal{D} = \mathcal{D}\left(G, (g_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta}\right)$$

for a (finite) abelian group G consists of elements $g_i \in G$, $\chi_j \in \widehat{G}$ and a Cartan matrix (a_{ij}) of finite type satisfying the Cartan condition

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}$$

with $q_{ii} \neq 1$, where $q_{ij} = \chi_j(g_i)$, in particular $q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}$ for all $1 \leq i, j \leq \theta$. In general, the matrix (q_{ij}) of a diagram of Cartan type is not symmetric, but by [AS, Lemma 1.2] it can be reduced to the symmetric case by twisting.

Let $\mathbf{Z}[I]$ be the free abelian group of rank θ with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_\theta\}$. The Weyl group $W \subset \mathrm{Aut}(\mathbf{Z}[I])$ of (a_{ij}) is generated by the reflections $s_i \colon \mathbf{Z}[I] \to \mathbf{Z}[I]$, where $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all i, j. The root system of the Cartan matrix (a_{ij}) is $\Phi = \cup_{i=1}^{\theta} W(\alpha_i)$ and $\Phi^+ = \Phi \cap \mathbf{Z}[I] = \left\{\alpha \in \Phi |\, \alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \geq 0\right\}$ is the set of positive roots relative to the basis of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_\theta\}$. Obviously, the number of positive roots $p = |\Phi^+|$ is at least θ. The maps $g \colon \mathbf{Z}[I] \to G$ and $\chi \colon \mathbf{Z}[I] \to \tilde{G}$ given by $g_\alpha = g_1^{n_1} g_2^{n_2} \ldots g_\theta^{n_\theta}$ and $\chi_\alpha = \chi_1^{n_1}\chi_2^{n_2}\ldots\chi_\theta^{n_\theta}$ for $\alpha =$

$\sum_{i=1}^{\theta} n_i\alpha_i$, respectively, are group homomorphisms. The bilinear map $q\colon \mathbf{Z}[I] \times \mathbf{Z}[I] \to k^\times$ defined by $q_{\alpha_i\alpha_j} = q_{ij}$ can be expressed as $q_{\alpha\beta} = \chi_\beta(g_\alpha)$.

If $\mathcal{X}$ the set of connected components of the Dynkin diagram of Φ let Φ_J be the root system of the component $J \in \mathcal{X}$. The partition of the Dynkin diagram into connected components corresponds to an equivalence relation on $I = \{1, 2, \dots, \theta\}$, where $i \sim j$ if α_i and α_j are in the same connected component.

Lemma 2.1. [AS, Lemma 2.3] *Suppose that $\mathcal{D}$ is a connected datum of finite Cartan type, i.e: the Dynkin diagram of the Cartan matrix (a_{ij}) is connected, and such that*

(1) *q_{ii} has odd order, and*
(2) *the order of q_{ii} is prime to 3, if (a_{ij}) is of type G_2.*

Then there are integers $d_i \in \{1, 2, 3\}$ for $1 \le i \le \theta$ and a $q \in k^\times$ of odd order N such that

$$q_{ii} = q^{2d_i} \quad . \quad d_i a_{ij} = d_j a_{ji}$$

for $1 \le i, j \le \theta$. If the Cartan matrix (a_{ij}) of $\mathcal{D}$ is of type G_2 then the order of q is prime to 3. In particular, the q_{ii} all have the same order in $k^\times$, namely N.

More generally, let $\mathcal{D}$ be a datum of finite Cartan type in which the order N_i of q_{ii} is odd for all i, and the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. It then follows that the order function N_i is constant, say equal to N_J, on each connected component J. A datum satisfying these conditions will be called special datum of finite Cartan type.

Fix a reduced decomposition of the longest element

$$w_0 = s_{i_1} s_{i_2} \dots s_{i_p}$$

of the Weyl group W in terms of the simple reflections. Then

$$\left\{s_{i_1} s_{i_2} \dots s_{i_{l-1}}(\alpha_{i_l})\right\}_{i=1}^{p}$$

is a convex ordering of the positive roots.

Let $V = V(\mathcal{D})$ be the crossed kG-module with basis $\{x_1, x_2, \dots, x_\theta\}$, where $x_i \in V_{g_i}^{\chi_i}$ for $1 \le i \le \theta$. Then for all $1 \le i \ne j \le \theta$ the elements $ad^{1-a_{ij}} x_i(x_j)$ are primitive in the free braided Hopf algebra $\mathcal{A}(V)$ (see Lemma 1.1 or [AS1, Appendix 1]). If $\mathcal{D}$ is as in the previous Lemma then $\chi_i^{1-a_{ij}}\chi_j \ne \varepsilon$. This implies that $f(u_{ij}) = 0$ for any braided (Hopf) subalgebra A of $\mathcal{A}(V)$ containing $u_{ij} = ad^{1-a_{ij}} x_i(x_j)$ and any G-invariant algebra map $f\colon A \to k$. Define root vectors in $\mathcal{A}(V)$ as follows by iterated braided commutators of the elements $x_1, x_2, \dots, x_\theta$, as in Lusztig's case but with the general braiding:

$$x_{\beta_l} = T_{i_1} T_{i_2} \dots T_{i_{l-1}}(x_{i_l}),$$

where $T_i(x_j) = \mathrm{ad}_{x_i}^{-a_{ij}}(x_j)$

In the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}} x_i(x_j) | 1 \le i \ne j \le \theta)$ define root vectors $x_\alpha \in \mathcal{A}(V)$ for $\alpha \in \Phi^+$ by the same iterated braided commutators of the elements $x_1, x_2, \dots, x_\theta$ as in Lusztig's case but with respect to the general

braiding. (See [AS2], and the inductive definition of root vectors in [Ri] or also [CP, Section 8.1 and Appendix].) Let $K(\mathcal{D})$ be the subalgebra of $R(\mathcal{D})$ generated by $\left\{ x_\alpha^N \,\middle|\, \alpha \in \Phi^+ \right\}$.

Theorem 2.2. [AS, Theorem 2.6] *Let $\mathcal{D}$ be a connected datum of finite Cartan type as in the previous Lemma. Then*

(1) $\left\{ x_{\beta_1}^{a_1} x_{\beta_2}^{a_2} \dots x_{\beta_p}^{a_p} \,\middle|\, a_1, a_2, \dots, a_p \geq 0 \right\}$ *forms a basis of* $R(\mathcal{D})$,

(2) $K(\mathcal{D})$ *is a braided Hopf subalgebra of* $R(\mathcal{D})$ *with basis*

$$\left\{ x_{\beta_1}^{Na_1} x_{\beta_2}^{Na_2} \dots x_{\beta_p}^{Na_p} \,\middle|\, a_1, a_2, \dots, a_p \geq 0 \right\},$$

(3) $[x_\alpha, x_\beta^N]_c = 0$, *i.e:* $x_\alpha x_\beta^N = q_{\alpha\beta}^N x_\beta^N x_\alpha$ *for all* $\alpha, \beta \in \Phi^+$.

The vector space $V = V(\mathcal{D})$ can also be viewed as a crossed module in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$. The Hopf algebra $\mathcal{A}(V)$, the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}} x_i(x_j) | 1 \leq i \neq j \leq \theta)$ and its Hopf subalgebra $K(\mathcal{D})$ generated by $\left\{ x_\alpha^N \,\middle|\, \alpha \in \Phi^+ \right\}$ are all Hopf algebras in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$. In particular, their comultiplications are $\mathbf{Z}[I]$-graded. By construction, for $\alpha \in \Phi^+$, the root vector $x_\alpha \in R(\mathcal{D})$ is $\mathbf{Z}[I]$-homogeneous of $\mathbf{Z}[I]$-degree α, so that $x_\alpha \in R(\mathcal{D})_{g_\alpha}^{\chi_\alpha}$. To simplify notation write for $1 \leq l \leq p$ and for $a = (a_1, a_2, \dots, a_p) \in \mathbf{N}^p$

$$h_l = g_{\beta_l}^N \ , \ \eta_l = \chi_{\beta_l}^N \ , \ z_l = x_{\beta_l}^N$$

and $\underline{a} = \sum_{i=1}^p a_i \beta_i$

$$h^a = h_1^{a_1} h_2^{a_2} \dots h_p^{a_p} \in G \ , \ \eta^a = \eta_1^{a_1} \eta_2^{a_2} \dots \eta_p^{a_p} \in \tilde{G} \ , \ z^a = z_1^{a_1} z_2^{a_2} \dots z_p^{a_p} \in K(\mathcal{D}).$$

In particular, for $e_l = (\delta_{kl})_{1 \leq k \leq p}$, where δ_{kl} is the Kronecker symbol, $\underline{e}_l = \beta_l$ and $z^{e_l} = z_l$ for $1 \leq l \leq p$. The height of $\alpha = \sum_{i=1}^\theta n_i \alpha_i \in \mathbf{Z}[I]$ is defined to be the integer $ht(\alpha) = \sum_{i=1}^\theta n_i$. Observe that if $a, b, c \in \mathbf{N}^p$ and $\underline{a} = \underline{b} + \underline{c}$ then

$$h^a = h^b h^c \ , \ \eta^a = \eta^b \eta^c \text{ and } \mathrm{ht}(\underline{b}) < \mathrm{ht}(\underline{a}) \text{ if } \underline{c} \neq 0.$$

The comultiplication on $K(\mathcal{D})$ is $\mathbf{Z}[I]$-graded, so that

$$\Delta_{K(\mathcal{D})}(z^a) = z^a \otimes 1 + 1 \otimes z^a + \sum_{b,c \neq 0; \underline{b}+\underline{c}=\underline{a}} t_{bc}^a z^b \otimes z^c$$

and hence

$$\Delta_{K(\mathcal{D})\# kG}(z^a) = z^a \otimes 1 + h^a \otimes z^a + \sum_{b,c \neq 0; \underline{b}+\underline{c}=\underline{a}} t_{bc}^a z^b h^c \otimes z^c$$

on the bosonization. The algebra $K(\mathcal{D})$ is generated by the subspace $L(\mathcal{D})$ with basis $\{z_1, z_2, \dots, z_p\}$. The (left) kG-module structure on $\mathcal{A}(V)$ restricts to $L(\mathcal{D})$, and induces (right) kG-actions on $\mathrm{Alg}(K(\mathcal{D}), k)$ and on $\mathrm{Vect}(L(\mathcal{D}), k)$ by the formula $(fg)(x) = f(gx)$. A linear functional $f \colon L(\mathcal{D}) \to k$ is called g-invariant if $fg = f$ for all $g \in G$. Let $\mathrm{Vect}_G(L(\mathcal{D}), k)$ be the subspace of G-invariant linear functionals in $\mathrm{Vect}(L(\mathcal{D}), k)$.

Proposition 2.3. *Let* $\mathrm{Vect}_G(L(\mathcal{D})$ *and* $\mathrm{Alg}_G(K(\mathcal{D}),k)$ *be the space of* G*-invariant linear functionals and the set of* G*-invariant algebra maps, where* $\mathcal{D}$ *is a connected special datum of finite Cartan type. Then:*

(1) $\mathrm{Vect}_G(L(\mathcal{D}),k) = \{ f \in \mathrm{Vect}(L(\mathcal{D}),k) | \, f(z_l) = 0 \text{ if } \eta_l \neq \varepsilon\}$.
(2) *The restriction map* $\mathrm{res}\colon \mathrm{Alg}_G(K(\mathcal{D}),k) \to \mathrm{Vect}_G(L(\mathcal{D}),k)$ *is a bijection. The inverse is given by* $\mathrm{res}^{-1}(f)(z^{\underline{a}}) = f(z_1)^{a_1} f(z_2)^{a_2} \dots f(z_p)^{a_p}$.
(3) $\mathrm{Alg}_G(K(\mathcal{D}),k)$ *is a group under convolution.*
(4) *The restriction map* $\mathrm{res}\colon {}_G\mathrm{Alg}_G(K(\mathcal{D})\#kG,k) \to \mathrm{Alg}_G(K(\mathcal{D}),k)$ *is an isomorphism of groups with inverse defined by* $\mathrm{res}^{-1}(f)(x\otimes g) = f(x)$, *and* ${}_G\,\mathrm{Alg}_G(K(\mathcal{D})\#kG,k) = \left\{ \tilde{f} \in \mathrm{Alg}(K(\mathcal{D})\#kG,k) \middle| \, \tilde{f}_{|kG} = \varepsilon \right\}$.
(5) *The map* $\Theta = \rho\,\mathrm{res}^{-1}\colon \mathrm{Alg}_G(K(\mathcal{D}),k) \to \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG)^{op}$, *defined by* $\Theta(f) = \mathrm{res}^{-1}(f) * 1 * \mathrm{res}^{-1}(f)s$, *is a group homomorphism whose image is a subgroup in*
$$\widetilde{\mathrm{Aut}}_{Hopf}(K(\mathcal{D})\#kG) = \left\{ f \in \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG) | \, f_{|kG} = id \right\}.$$
(6) *For every* $f \in \mathrm{Alg}_G(K(\mathcal{D}),k)$ *the automorphism* $\Theta(f)$ *of* $K(\mathcal{D})\#kG$ *is determined by*
$$\Theta(f)z^a = z^a + f(z^a)(1-h^a) + \sum_{b,c\neq 0;\underline{b}+\underline{c}=\underline{a}} t^a_{bc} f(z^b)z^c$$
$$+ \sum_{b,c\neq 0;\underline{b}+\underline{c}=\underline{a}} t^a_{bc} \left[z^b + f(z^b)(1-h^b) + \sum_{d,e\neq 0;\underline{d}+\underline{e}=\underline{b}} t^b_{d,e} f(z^d) z^e \right] h^c f s(z^c).$$
In particular, $\Theta(f)z^a = z^a + f(z^a)(1-h^a)$ *if* $ht(\underline{a}) = 1$.

Proof. If $f \in \mathrm{Vect}_G(L(\mathcal{D}),k)$ then $f(z_i) = f(gz_i) = \eta_i(g)f(z_i)$ for all $1 \leq i \leq p$ and for all $g \in G$. Thus, $f(z_i) = 0$ if $\eta_i \neq \varepsilon$.

By Theorem 2.2 it follows that
$$K(\mathcal{D}) \cong TL(\mathcal{D})/(z_iz_j - \eta_j(h_i)z_jz_i | 1 \leq i < j \leq p).$$
If $f \in \mathrm{Vect}_G(L(\mathcal{D}),k)$ then the induced algebra map $\tilde{f}\colon TL(\mathcal{D}) \to k$ factors uniquely through $K(\mathcal{D})$, since
$$\tilde{f}(z_iz_j - \eta_j(h_i)z_jz_i) = f(z_i)f(z_j) - \eta_j(h_i)f(z_j)f(z_i) = f(z_i)(f(z_j) - f(h_iz_j)) = 0$$
for $1 \leq i, j \leq p$, by the fact that f is G-invariant. This proves the second assertion.

The next three assertions are a special case of 1.6.

The set of all algebra maps $\mathrm{Alg}(K(\mathcal{D}),k)$ may not be a group under convolution, but the subset $\mathrm{Alg}_G(K(\mathcal{D}),k)$ is. If f_1, f_2 and f are G-invariant then
$$\begin{aligned} f_1 * f_2(xy) &= (f_1\otimes f_2)(m\otimes m)(1\otimes c\otimes 1)(x_1\otimes x_2\otimes y_1\otimes y_2) \\ &= (f_1\otimes f_2)(x_1(x_2)_{-1}y_1 \otimes (x_2)_0 y_2 \\ &= f_1(x_1)\varepsilon((x_2)_{-1})f_1(y_1)f_2((x_2)_0)f_2(y_2) \\ &= f_1(x_1)f_1(y_1)f_2(x_2)f_2(y_2) = f_1 * f_2(x) f_1 * f_2(y) \end{aligned}$$

and moreover, $(f_1*f_2)g = f_1g*f_2g = f_1*f_2$, $fsg = fgs = fs$, $\varepsilon*f = f = f*\varepsilon$ and $f*fs = \varepsilon = fs*f$ so that $\mathrm{Alg}_G(K(\mathcal{D})$ is closed under convolution multiplication and inversion.

The map $\Psi\colon \mathrm{Alg}_G(K(\mathcal{D}),k) \to \mathrm{Alg}(K(\mathcal{D})\#kG,k)$ given by $\Psi(f)(x\otimes g) = f(x)$, is a homomorphism, since

$$\begin{aligned}\Psi(f_1)*\Psi(f_2)(x\otimes g) &= \Psi(f_1)\otimes\Psi(f_2)(x_1\otimes(x_2)_{-1}g\otimes(x_2)_0\otimes g)\\ &= f_1(x_1)\varepsilon((x_2)_{-1})f_2((x_2)_0)\\ &= f_1(x_1)f_2(x_2) = f_1*f_2(x)\\ &= \Psi(f_1*f_2)(x\otimes g).\end{aligned}$$

The inverse $\Psi^{-1}\colon {}_G\mathrm{Alg}_G(K(\mathcal{D})\#kG,k) \to \mathrm{Alg}_G(K(\mathcal{D}),k)$, given by $\Psi^{-1}(\tilde{f})(x) = \tilde{f}(x\otimes 1)$, is just the restriction map.

It is convenient to use the notation $\Psi(f) = \tilde{f}$. Then $\Theta(f) = \tilde{f}*1*\tilde{f}s$ and

$$\begin{aligned}\Theta(f_1*f_2) &= \widetilde{f_1*f_2}*1*\widetilde{f_1*f_2}s\\ &= (\tilde{f}_1*\tilde{f}_2)*1*(\tilde{f}_2s*\tilde{f}_1s)\\ &= \Theta(f_2)\Theta(f_1).\end{aligned}$$

In particular, $\Theta(f)\Theta(fs) = \Theta(fs*f) = \Theta(\varepsilon) = 1 = \Theta(f*fs) = \Theta(fs)\Theta(f)$. Moreover,

$$\begin{aligned}\Theta(f)(xy) &= \tilde{f}(x_1y_1)x_2y_2\tilde{f}s(x_3y_3)\\ &= \tilde{f}(x_1)\tilde{f}(y_1)x_2y_2\tilde{f}s(y_3)\tilde{f}s(x_3)\\ &= \tilde{f}(x_1)x_2\tilde{f}s(x_3)\tilde{f}(y_1)y_2\tilde{f}s(y_3)\\ &= \Theta(f)(x)\Theta(f)(y)\end{aligned}$$

and

$$\begin{aligned}\Delta\Theta(f) &= \Delta(\tilde{f}*1*\tilde{f}s)\\ &= \Delta(\tilde{f}\otimes 1\otimes\tilde{f}s)\Delta^{(2)}\\ &= (\tilde{f}\otimes 1\otimes 1\otimes\tilde{f}s)\Delta^{(3)}\\ &= (\tilde{f}\otimes 1\otimes\varepsilon\otimes 1\otimes\tilde{f}s)\Delta^{(4)}\\ &= (\tilde{f}\otimes 1\otimes\tilde{f}s\otimes\tilde{f}\otimes 1\otimes\tilde{f}s)\Delta^{(5)}\\ &= (\tilde{f}*1*\tilde{f}s\otimes\tilde{f}*1*\tilde{f}s)\Delta\\ &= (\Theta(f)\otimes\Theta(f))\Delta,\end{aligned}$$

showing that $\Theta(f)$ is an automorphism of $K(\mathcal{D})\#kG$ with inverse $\Theta(fs)$.

The remaining item now follows from the formula for the comultiplication

$$\Delta(z^a) = z^a\otimes 1 + h^a\otimes z^a + \sum_{b,c\neq 0;\underline{b}+\underline{c}=\underline{a}} t^a_{bc}z^bh^c\otimes z^c$$

of $K(\mathcal{D})\#kG$, which implies

$$\Delta^{(2)}(z^a) = z^a \otimes 1 \otimes 1 + h^a \otimes z^a \otimes 1 + \sum t^a_{bc} z^b h^c \otimes z^c \otimes 1 + h^a \otimes h^a \otimes z^a$$
$$+ \sum t^a_{bc} \left[z^b h^c \otimes h^c \otimes z^c + h^a \otimes z^b h^c \otimes z^c + \sum t^b_{rl} z^r h^{l+c} \otimes z^l h^c \otimes z^c \right].$$

and

$$1 * s(z^a) = z^a + h^a s(z^a) + \sum t^a_{bc} z^b h^c s(z^c) = \varepsilon(z^a) = 0.$$

Applying $\Psi(f) = \tilde{f}$ to the latter gives

$$f(z^a) + fs(z^a) + \sum t^a_{bc} f(z^b) fs(z^c) = 0,$$

which will be used in the following evaluation. Now compute

$$\begin{aligned}
\Theta(f)(z^a) \quad &= \quad f(z^a) + z^a + \sum t^a_{bc} f(z^b) z^c + h^a fs(z^a) \\
&\quad\quad + \sum t^a_{bc} [f(z^b) + z^b + \sum t^b_{rl} f(z^r) z^l] h^c fs(z^c) \\
&= \quad z^a + f(z^a)(1 - h^a) + \sum t^a_{bc} f(z^b) z^c \\
&\quad\quad + \sum t^a_{bc} [z^b + f(z^b)(1 - h^b) + \sum t^b_{rl} f(z^r) z^l] h^c fs(z^c)
\end{aligned}$$

to get the required result. □

For any $f \in \mathrm{Alg}_G(K(\mathcal{D}), k)$ define by induction on $ht(\underline{a})$ the following elements in the augmentation ideal of kG

$$u_a(f) = f(z^a)(1 - h^a) + \sum_{b,c \neq 0; \underline{b} + \underline{c} = \underline{a}} t^a_{bc} f(z^b) u_c(f),$$

where $u_a(f) = f(z^a)(1 - h^a)$ if $ht(\underline{a}) = 1$. In particular, for a positive root $\alpha = \beta_l \in \Phi^+$ and $x^N_\alpha = x^N_{\beta_l} = z^{\underline{e}_l}$ write $u_l(f) = u_{\underline{e}_l}(f) = u_\alpha(f)$. We can think of $f = (f(x^N_\alpha) | \alpha \in \Phi^+)$ as root vector parameters in the sense of [AS].

Corollary 2.4. *Let $\mathcal{D}$ be a special connected datum of finite Cartan type. Then*

$$u(\mathcal{D}, f) = R(\mathcal{D})\#kG/(x^N_\alpha + u_\alpha(f))$$

are the liftings of $\mathcal{B}(V)\#kG = u(\mathcal{D}, \varepsilon)$.

Proof. The augmentation ideal of $K(\mathcal{D})$, the ideal I of $K(\mathcal{D})\#kG$ and the ideal (I) in $R(\mathcal{D})\#kG$ generated by $\left\{ x^N_\alpha \middle| \alpha \in \mathcal{X} \right\}$ are Hopf ideals. It follows from the inductive formulas for $\Theta(f)(z^a)$ and $u_a(f)$ above that for every $f \in \mathrm{Alg}_G(K(\mathcal{D}), k)$ the ideals $I_f = \Theta(f)(I)$ in $K(\mathcal{D})\#kG$ and (I_f) in $R(\mathcal{D})\#kG$ generated by $\left\{ x^N_\alpha + u_\alpha(f) \middle| \alpha \in \Phi^+ \right\}$ are Hopf ideals as well. The Hopf algebras $u(\mathcal{D}, f) = R(\mathcal{D})\#kG/(\Theta(f)(I))$ are the liftings of $u(\mathcal{D}, \varepsilon) = \mathcal{B}(V)\#kG$ parameterized by $f = (f(x^N_\alpha | \alpha \in \Phi^+) \in \mathrm{Alg}_G(K(\mathcal{D}), k)$. □

In the not necessarily connected case of a special datum of finite Cartan type the elements $ad^{1-a_{ij}}x_i(x_j)$ are still primitive in $\mathcal{A}(V)$ and $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}}x_i(x_j)|i \sim j)$ is still a Hopf algebra, which contains $R(\mathcal{D}_J)$ for every connected component $J \in \mathcal{X}$. The Hopf subalgebra $K(\mathcal{D})$ generated by the subspace with basis $S = \{ z_J^{a_J}, z_{ij} | \, J \in \mathcal{X}, i \not\sim j\}$, where $z_{ij} = [x_i, x_j]_c$, contains $K(\mathcal{D}_J)$ for every $J \in \mathcal{X}$. The comultiplication in the components $K(\mathcal{D}_J)$ and $K(\mathcal{D}_J)\#kG$ is of course given as before in the connected case, while for $i \not\sim j$

$$\Delta(z_{ij}) = z_{ij} \otimes 1 + 1 \otimes z_{ij}$$

in $K(\mathcal{D})$ and $R(\mathcal{D})$ and

$$\Delta(z_{ij}) = z_{ij} \otimes 1 + g_i g_j \otimes z_{ij}$$

in the bozonizations $K(\mathcal{D})\#kG$ and $R(\mathcal{D})\#kG$. The space of G-invariant linear functionals $\mathrm{Vect}_G(L(\mathcal{D}), k)$ consists of the elements $f \in \mathrm{Vect}(L(\mathcal{D}), k)$ such that

$$f(z_r) = 0 \text{ if } \eta_r \neq \varepsilon \text{ for } 1 \leq r \leq p \text{ and } f(z_{ij}) = 0 \text{ if } \chi_i\chi_j \neq \varepsilon \text{ if } i \not\sim j.$$

The induced algebra map $\tilde{f}\colon TL(\mathcal{D}) \to k$ of such a linear functional satisfies

- $\tilde{f}([z_r, z_s]_c) = f(z_r)(f(z_s) - f(h_r z_s)) = 0,$
- $\tilde{f}([z_{ij}, z_r]_c) = f(z_{ij})(f(z_r) - f(g_i g_j z_r) = 0,$
- $\tilde{f}([z_{ij}, z_{lm}]_c) = f(z_{ij})(f(z_{lm}) - f(g_i g_j z_{lm})) = 0,$

since f is G-invariant. It therefore factors through $K(\mathcal{D})$, since

$$TL(\mathcal{D})/([z_r, z_s]_c, [z_{ij}, z_r]_c, [z_{ij}, z_{lm}]_c) = K(\mathcal{D})/([z_r, z_s]_c, [z_{ij}, z_r]_c, [z_{ij}, z_{lm}]_c).$$

It follows that the restriction maps

$$\mathrm{res}\colon {}_G\,\mathrm{Alg}_G(K(\mathcal{D})\#kG, k) \to \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Vect}_G(L(\mathcal{D}), k)$$

are bijective, and $f = \{ f(z_{ij}) | \, i \not\sim j\} \cup \{ f(z_r) | \, 1 \leq r \leq p\}$ can be interpreted as a combination of linking parameters and root vector parameters in the sense of [AS]. Then map

$$\Theta\colon \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG)^{op}$$

given by $\Theta(f) = f * 1 * fs$ is a homomorphism of groups. Moreover, since $z_{ij} + g_i g_j s(z_{ij}) = m(1 \otimes s)\Delta(z_{ij}) = 0$ in $K(\mathcal{D})\#kG$, it follows that

$$\Theta(f)(z_{ij}) = (f \otimes 1 \otimes fs)\Delta^{(2)}(z_{ij}) = z_{ij} + f(z_{ij})(1 - g_j g_j)$$

when $i \not\sim j$, while $\Theta f(z_r)$ is given inductively as in 2.3. In this way one obtains therefore all the 'liftings' of $B(V)\#kG$ for special data of finite Cartan type.

Theorem 2.5. *Let $\mathcal{D}$ be a special datum of finite Cartan type. Then*

$$u(\mathcal{D}, f) = R(\mathcal{D})\#kG/(x_\alpha^{N_\alpha} + u_\alpha(f), [x_i, x_j]_c + f(z_{ij})(1 - g_i g_j)|\alpha \in \Phi^+, i \not\sim j)$$

for $f \in \mathrm{Vect}_G(L(\mathcal{D}), k)$ are the liftings of $\mathcal{B}(V)\#kG = u(\mathcal{D}, \varepsilon)$. Moreover, all these liftings are monoidally Morita-Takeuchi equivalent.

Proof. Clearly, $u(\mathcal{D}, f)$ is a lifting of $\mathcal{B}(V)\#kG$ for the root vector parameters $\{\mu_\alpha = f(x_\alpha^{N_\alpha}) | \alpha \in \Phi^+\}$ and the linking parameters $\{\lambda_{ij} = f([x_i, x_j]_c) | i \not\sim j\}$. By [AS] all liftings of $\mathcal{B}(V)\#kG$ are of that form. To proof the last assertion use 1.5 with $H = kG$, $K = K\mathcal{D}$, $f \in \mathrm{Alg}_G(K, k)$. Then the ideal $I = (x_\alpha^{n_\alpha}, [x_i, x_j]_c | \alpha \in \Phi^+, i \not\sim j)$ and $J = \Theta(f)(I) = (x_\alpha^{N_\alpha} + u_\alpha(f), [x_i, x_j]_c + f(z_{ij})(1 - g_i g_j) | \alpha \in \Phi^+, i \not\sim j)$ of $K\#kG$ are conjugate. By 1.5 the quotient Hopf algebras $u(\mathcal{D}, \varepsilon)$ and $u(\mathcal{D}, f)$ of $R(\mathcal{D})\#kG$ are monoidally Morita-Takeuchi equivalent. The additional condition $(R\#kG)/(\mathrm{res}^{-1}(f) * (I\#kG)) \neq 0$ is verified in ([Ma1], Appendix). $\square$

3. Main Results

In this section we describe liftings of special crossed modules V over finite abelian groups in terms of cocycle deformations of $B(V)\#kG$.

A normalized 2-cocycle $\sigma\colon A \otimes A \to k$ on a Hopf algebra A is a convolution invertible linear map such that

$$(\varepsilon \otimes \sigma) * \sigma(1 \otimes m) = (\sigma \otimes \varepsilon) * \sigma(m \otimes 1)$$

and $\sigma(\iota \otimes 1) = \varepsilon = \sigma(1 \otimes \iota)$. The deformed multiplication

$$m_\sigma = \sigma * m * \sigma^{-1}\colon A \otimes A \to A$$

and antipode

$$s_\sigma = \sigma * s * \sigma^{-1}\colon A \to A$$

on A, together with the original unit, counit and comultiplication define a new Hopf algebra structure on H which we denote by A_σ.

Theorem 3.1. [Sch, Corollary 5.9] *If two Hopf algebras A and A' are cocycle deformations of each other, then they are monoidally Morita-Takeuchi equivalent. The converse is true if A and A' are finite dimensional.*

Suppose now that V is a crossed kG-module of special finite Cartan type, $\mathcal{A}(V)$ the free braided algebra and $\mathcal{A}(V)\#kG$ its bosonization. If I is the ideal of $\mathcal{A}(V)$ generated by the subset

$$S = \{ ad^{1-a_{ij}} x_i(x_j) | i \sim j\} \cup \{ x_\alpha^{N_\alpha} | \alpha \in \Phi^+\} \cup \{ [x_i, x_j]_c | i \not\sim j\}$$

then $\mathcal{A}(V)/I = \mathcal{B}(V)$ is the Nichols algebra. The subalgebra K of $\mathcal{A}(V)$ generated by S is a Hopf subalgebra [AS], [CP, Proposition 9.2.1]. Then $K\#kG$ is the Hopf subalgebra of $\mathcal{A}(V)\#kG$ generated by S and G.

Lemma 3.2. *The injective group homomorphism*

$$\phi\colon \mathrm{Alg}_G(K, k) \to \mathrm{Alg}(K\#kG, k)$$

given by $\phi(f)(x\#g) = f(x)$ has image

$${}_G\mathrm{Alg}_G(K\#kG, k) = \{ f \in \mathrm{Alg}(K\#kG, k) | f_{|kG} = \varepsilon\}$$

and

$$\mathrm{adj}\colon \mathrm{Alg}_G(K, k) \to \mathrm{Aut}(K\#kG)^{op}$$

has its image in the subgroup

$$\widetilde{Aut(K\#kG)} = \left\{ f \in \mathrm{Aut}(K\#kG) |\, f_{|kG} = \varepsilon \right\}.$$

Moreover, if V *is of special finite Cartan type then* $f(ad^{1-a_{ij}}x_i(x_j)) = 0$ *for* $i \sim j$ *and for every* $f \in \mathrm{Alg}_G(K, k)$.

Proof. If $f \in \mathrm{Alg}(K\#kG, k)$ then $f(ad^{1-a_{ij}}x_i(x_j)) = f(g \cdot ad^{1-a_{ij}}x_i(x_j)g^{-1}) = \chi_i(g)^{1-a_{ij}}\chi_j(g)$, $f(x_\alpha^N) = f(gx_\alpha^{N_\alpha}g^{-1}) = \chi_\alpha^{N_\alpha}(g)f(x_\alpha^N)$ and $f([x_i, x_j]_c) = f(g[x_i, x_j]_c g^{-1}) = \chi_i(g)\chi_j(g)f([x_i, x_j]_c)$, so that $f(g \cdot ad^{1-a_{ij}}x_i(x_j)) = 0$ if $\chi_i^{1-a_{ij}}\chi_j \neq 0$, $f(x_\alpha^{N_\alpha}) = 0$ if $\chi_\alpha^{N_\alpha} \neq \varepsilon$ and $f([x_i, x_j]_c) = 0$ if $\chi_i\chi_j \neq \varepsilon$. □

The theorem above can now be applied to the situation in Section 3 to show that all 'liftings' of a crossed kG-module of special finite Cartan type are cocycle deformations of each other. The special case of quantum linear spaces has been studied by Masuoka [Ma], and that of a crossed kG-module corresponding to a finite number of copies of type A_n by Didt [Di].

Theorem 3.3. *Let* G *be a finite abelian group,* V *a crossed* kG*-module of special finite Cartan type,* $\mathcal{B}(V)$ *its Nichols algebra with bosonization* $A = \mathcal{B}(V)\#kG$. *Then:*

(1) *All liftings of* A *are monoidally Morita-Takeuchi equivalent, i.e: their comodule categories are monoidally equivalent, or equivalently,*
(2) *all liftings of* A *are cocycle deformations of each other.*

Proof. Theorem 2.5 at the end the last section says that $\mathcal{B}(V)\#kG \cong u(\mathcal{D}, \varepsilon) \cong R(\mathcal{D})\#kG/(I)$ for a Hopf ideal I in the Hopf subalgebra $K(\mathcal{D})\#kG$ of $R(\mathcal{D})\#kG$, that its liftings are of the form $u(\mathcal{D}, f) \cong R(\mathcal{D})\#kG/(I_f)$ for a conjugate Hopf ideal I_f, where $f \in \mathrm{Alg}_G(K(\mathcal{D}, k), k) \cong {}_G\,\mathrm{Alg}_G(K(\mathcal{D})\#kG, k)$, and that they are all Morita-Takeuchi equivalent. Thus, Schauenburg's result applies, so that all these liftings are cocycle deformations of each other. □

Corollary 3.4. *Let* H *be a finite dimensional pointed Hopf algebra with abelian group of points* $G(H) = G$ *and assume that the order of* G *has no prime divisors* < 11. *Then:*

- H *and* $\mathrm{gr}_c(H)$ *are Morita Takeuchi equivalent, or equivalently,*
- H *is a cocycle deformation of* $\mathrm{gr}_c(H)$.
- *There are only finitely many quasi-isomorphism classes of such Hopf algebras in any given dimension.*

Proof. Under the present assumptions the Classification Theorem [AS] asserts that $\mathrm{gr}_c(H) \cong B(V)\#kG$ for a crossed kG-module V of special finite Cartan type, and hence the Theorem 3.3 applies. Moreover, there are only finitely many isomorphism classes of Hopf algebras of the form $B(V)\#kG$ in any given dimension. □

4. About the deforming cocycles

4.1. Deforming cocycles and their infinitesimal parts. Let $A = \bigoplus_{n=1}^{\infty} A_n$ be a $\mathbb{N}$-graded Hopf algebra (here $\mathbb{N} = \{0, 1, 2 \ldots\}$, and A_n denotes the homogeneous component of A of degree n). Let $\sigma\colon A \otimes A \to k$ be a normalized 2-cocycle as discussed at the beginning of Section 3. *Throughout the Section we additionally assume that*

$$\sigma|_{A_0 \otimes A_0} = \varepsilon.$$

We decompose $\sigma = \sum_{n=0}^{\infty} \sigma_i$ into the (locally finite) sum of homogeneous maps σ_i od degree $-i$; more precisely

$$\sigma_i = A \otimes A \to \bigoplus_{p+q=i} A_p \otimes A_q = (A \otimes A)_i \overset{\sigma|_{(A\otimes A)_i}}{\to} k.$$

Note that due to our assumption $\sigma|_{A_0 \otimes A_0} = \varepsilon$ we have

$$\sigma_0 = \varepsilon.$$

In this fashion we also decompose $\sigma^{-1} = \sum_{j=0}^{\infty} \eta_j$, that is η_j's are homogeneous maps uniquely determined by $\displaystyle\sum_{i+j=\ell} \sigma_i \eta_j = \delta_{\ell,0}$ for $\ell = 1, 2 \ldots$ and automatically also satisfy $\displaystyle\sum_{i+j=\ell} \eta_i \sigma_j = \delta_{\ell,0}$. Note that $\eta_0 = \varepsilon$ and that for the least positive integer s for which $\sigma_s \neq 0$ we have $\eta_s = -\sigma_s$.

The cocycle condition

$$(\varepsilon \otimes \sigma(t)) * \sigma(t)(1 \otimes m) = (\sigma(t) \otimes \varepsilon) * \sigma(t)(m \otimes 1)$$

implies that

$$\sum_{i+j=\ell} (\varepsilon \otimes \sigma_i) * \sigma_j(1 \otimes m) = \sum_{i+j=\ell} (\sigma_i \otimes \varepsilon) * \sigma_j(m \otimes 1)$$

for all $\ell \geq 1$. In particular, if s is the least positive integer for which $\sigma_s \neq 0$, then

$$\varepsilon \otimes \sigma_s + \sigma_s(1 \otimes m) = \sigma_s \otimes \varepsilon + \sigma_s(m \otimes 1)$$

so that

$$\sigma_s\colon A \otimes A \to k$$

is a *Hochschild 2-cocycle*. We call this Hochschild 2-cocycle σ_s the *graded infinitesimal part* of σ.

4.2. **Relationship with formal graded deformations.** The cocycle deformation A_σ is a filtered bialgebra with the underlying filtration inherited from the grading on A, i.e., the ℓ-th filtered part is $(A_\sigma)_{(\ell)} = \bigoplus_{i=0}^{\ell} A_i$. Note that the associated graded bialgebra GrA_σ can be identified with A. It was observed by Du,Chen, and Ye [DCY] that decomposing the multiplication $m_\sigma = \sigma * m * \sigma^{-1}$ into the sum of homogeneous components m_i of degree $-i$, allows us to identify the filtered k-linear structure A_σ with a $k[t]$-linear structure $m^t_\sigma \colon A[t] \otimes A[t] \to A[t]$ induced by $m^t_\sigma|_{A\otimes A} = m + m_1 t + m_2 t^2 + \dots$ (here we assume that the degree of t is 1). If $\Delta^t \colon A[t] \to A[t] \otimes A[t]$ is the $k[t]$-linear map induced by $\Delta^t|_A = \Delta$, then the graded Hopf algebra $A[t]_\sigma = (A[t], m^t_\sigma, \Delta^t)$ is a formal graded deformation of A in the sense of Du, Chen, and Ye [DCY] (see also [GS, MW]). Note that in case m does not commute with the graded infinitesimal part σ_s of σ, then $(m_s, 0)$, where $m_s = \sigma_s * m - m * \sigma_s$, is the infinitesimal part of the formal graded deformation (and is a 2-cocycle in the graded version of the truncated Gerstenhaber-Schack bialgebra cohomology).

4.3. **Exponential Map.** It is in general very hard to give explicit examples of multiplicative cocycles. One somewhat accessible family consists of bicharacters. Below we give another idea which can sometimes be used.

Note that if $A = \oplus_{n=0}^{\infty} A_n$ is a graded bialgebra, and $f \colon A \to k$ is a linear map such that $f|_{A_0} = 0$, then

$$e^f = \sum_{i=0}^{\infty} \frac{f^{*i}}{i!} \colon A \to k$$

is a well defined convolution invertible map with convolution inverse e^{-f}. When $f \colon A \otimes A \to k$ is a Hochschild cocycle (more precisely, we have $\varepsilon(a) f(b,c) + f(a, bc) = f(a,b)\varepsilon(c) + f(ab, c)$ and for all $a, b, c \in A$) such that $f|_{A\otimes A_0 + A_0 \otimes A} = 0$, then 'often' $e^f \colon A \otimes A \to k$ will be a multiplicative cocycle. For instance this happens whenever $f(1\otimes m)$ and $f(m\otimes 1)$ commute (with respect to the convolution product) with $\varepsilon \otimes f$ and $f \otimes \varepsilon$, respectively. Also note that if $f * f = 0$, then $e^f = \varepsilon + f$.

Lemma 4.1. *If $f \colon A \otimes A \to k$ is a Hochschild 2-cocycle such that $f(1 \otimes m)$ commutes with $\varepsilon \otimes f$ and $f(m \otimes 1)$ commutes with $f \otimes \varepsilon$ in the convolution algebra $\mathrm{Hom}_k(A\otimes A\otimes A, k)$, then e^f is a (multiplicative) 2-cocycle with graded infinitesimal part equal to f.*

Proof. Since f is a Hochschild 2-cocycle we have $\varepsilon\otimes f + f(1\otimes m) = f\otimes\varepsilon + f(m\otimes 1)$. Note that $e^{\varepsilon\otimes f} = \varepsilon \otimes e^f$ and $e^{f\otimes\varepsilon} = e^f \otimes \varepsilon$. Since m is a coalgebra map we also have $e^{f(m\otimes 1)} = e^f(m\otimes 1)$ and $e^{f(1\otimes m)} = e^f(1\otimes m)$. Also recall that for commuting

$gandh$ we have $e^{g+h} = e^g * e^h$. Hence we have

$$\begin{aligned}(\varepsilon\otimes e^f) * \left(e^f(1\otimes m)\right) &= e^{\varepsilon\otimes f} * e^{f(1\otimes m)}\\ &= e^{\varepsilon\otimes f + f(1\otimes m)}\\ &= e^{f\otimes\varepsilon + f(m\otimes 1)}\\ &= e^{f\otimes\varepsilon} * e^{f(m\otimes 1)}\\ &= (e^f\otimes\varepsilon) * \left(e^f(m\otimes 1)\right).\end{aligned}$$

□

4.4. **Exponential map in a quantum linear space.** In the rest we will study $u(\mathcal{D}, f)$ where the Dynkin diagram associated to $\mathcal{D}$ is of type $A_1 \times \ldots \times A_1$. We study the linking and root-vector parts of f separately and use notation $u(\mathcal{D}, f) = u(\mathcal{D}, \lambda, \mu)$, where $\mu = (\mu_\alpha)_{\alpha\in\Phi^+} = (f(x_\alpha^{N_\alpha}))_{\alpha\in\Phi^+}$ (root-vector parameters), and $\lambda = (\lambda_{i,j})_{1\le i<j<\theta} = (f([x_i, x_j]_c))_{1\le i<j\le\theta}$ (linking parameters).

From now on assume that $A = u(\mathcal{D}, 0, 0)$ is obtained as the bosonization of a quantum linear space. More precisely,

$$A = B(V)\#kG = \left\langle G, x_1, \ldots, x_\theta \,|\, gx_i = \chi_i(g)x_i g, x_i x_j = \chi_j(g_i)x_j x_i, x_i^{N_i} = 0\right\rangle.$$

Here $V = \oplus_{i=1}^{\theta} kx_i$, and $\chi_1, \ldots, \chi_\theta \in \widehat{G}$, $g_1, \ldots, g_\theta \in G$ are such that $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \ne j$, and the number N_i is the order of $\chi_i(g_i)$. Recall that $q_{ij} = \chi_i(g_j)$, and let $e_i = (\delta_{ij})_{j=1}^{\theta}$ for $1 \le i \le \theta$, where δ_{ij} is the Kronecker symbol. Then $\zeta_i\colon A\otimes A \to k$, given by

$$\zeta_i(x^a g, x^b h) = \begin{cases}\chi_i^{b_i}(g), & \text{if } a = a_i e_i, b = b_i e_i, a_i + b_i = N_i\\ 0, & \text{otherwise}\end{cases}$$

for $x^a = x_1^{a_1}\ldots x_\theta^{a_\theta}$ and $x^b = x_1^{b_1}\ldots x_\theta^{b_\theta}$ (see for example [MW]) is a Hochschild cocycle. Moreover, each of the sets

$$S_l = \{\, (\varepsilon\otimes\zeta_i), \zeta_i(1\otimes m) |\, 1 \le i \le \theta\}$$

and

$$S_r = \{\, (\zeta_i\otimes\varepsilon), \zeta_i(m\otimes 1) |\, 1 \le i \le \theta\}$$

is a commutative set (for the convolution product). We sketch the proof for S_l (the proof for S_r is symmetric). The maps $\zeta_i(1\otimes m)$ and $\zeta_j(1\otimes m)$ commute since ζ_i and ζ_j do. The same goes for $\varepsilon\otimes\zeta_i$ and $\varepsilon\otimes\zeta_j$. Hence it is sufficient to prove that for all i, j we have

$$(\varepsilon\otimes\zeta_i) * (\zeta_j(1\otimes m)) = (\zeta_j(1\otimes m)) * (\varepsilon\otimes\zeta_i).$$

If $i \ne j$, this is immediate. For $i = j$ note that both left and right hand side can be nonzero only at PBW elements of the form $x_i^r f \otimes x_i^s g \otimes x_i^p h \in A\otimes A\otimes A$, with

$r+s+p=2N_i$. Without loss of generality assume that $f=g=h=1$. In this case the left hand side evaluates to

$$\sum_{u+v=N_i}\binom{s}{u}_{q_{ii}}\binom{p}{v}_{q_{ii}}q_{ii}^{u(p-v)}=1$$

and the right hand side is

$$\sum_{u+v=N_i}\binom{r}{u}_{q_{ii}}\binom{s}{v}_{q_{ii}}q_{ii}^{u(s-v)}=1.$$

Thus we have the following.

Proposition 4.2. *If $\zeta=\sum_i\mu_i\zeta_i$, (where ζ_i's are as above), then $\zeta\colon A\otimes A\to k$ is a Hochschild 2-cocycle, and $\sigma=e^{\zeta}$ is a multiplicative cocycle. The associated cocycle deformation is a lifting of A with presentation*

$$\begin{aligned} A_\sigma &= u(\mathcal{D},0,\mu)\\ &= \left\langle G,x_1,\dots,x_\theta \,|\, gx_i=\chi_i(g)x_ig, x_ix_j=\chi_j(g_i)x_jx_i, x_i^{N_i}=\mu_i(1-g_i^{N_i})\right\rangle. \end{aligned}$$

Proof. Note that for $1\le i<j\le\theta$ we have $m_{A_\sigma}(x_i\otimes x_j-\chi_j(g_i)x_j\otimes x_i)=0$ and $m_{A_\sigma}(x_i^{\otimes N_i})=\mu_i(1-g_i^{N_i})$. Hence A is a quotient of $u(\mathcal{D},0,\mu)$. Since the dimensions of A_σ and $u(\mathcal{D},0,\mu)$ are the same we conclude that $A_\sigma=u(\mathcal{D},0,\mu)$. □

4.5. **q-exponential map.** Here we use the ideas and results on q-exponential maps of Sarah Witherspoon [W] in order to address the linking cocycles. The second author thought of this idea in a conversation with Margaret Beattie about the papers [GM1] and [ABM]. The first author obtained a similar idea independently and earlier through cohomological considerations [Gr2].

If q is a root of 1 and x is an element of any algebra B, then we can define a q-exponential function as follows.

Definition 4.3. *If $\ell\ge 2$ and q is a primitive ℓ-th root of unity then* $\exp_q(x)=\sum_{m=0}^{\ell-1}\frac{1}{(m)!_q}x^m$.

We will need the following nice addition formula proved in [W].

Lemma 4.4 ([W]). *If $\ell\ge 2$, q is a primitive ℓ-th root of unity, $yx=qxy$, and $x^iy^{\ell-i}=0$ for $i=0,\dots,\ell$, then* $\exp_q(x+y)=\exp_q(x)\exp_q(y)$.

From now on assume, as in the subsection above, that A is a bosonization of a quantum linear space. Extend the characters χ_i to all of A by letting

$$\chi_i(xg)=\varepsilon(x)\chi_i(g)$$

for any PBW-element $x\in A$. Define skew-derivations $d_i\colon A\to k$ by setting

$$d_i((x_jg)=\delta_{ij},$$

so that $\chi_i m = \chi_i \otimes \chi_i$ and $d_i m = d_i \otimes \varepsilon + \chi_i \otimes d_i$. Consider the linear maps

$$\eta_{j,i} = d_j * \chi_i \otimes d_i \colon A \otimes A \to k.$$

for which we have the following result.

Remark 4.5. *If i, j are linkable roots (i.e., $\chi_i\chi_j = \varepsilon$, $g_i g_j \neq 1$, and $\chi_i(g_j)\chi_j(g_i) = 1$), then $\eta_{j,i} = d_j * \chi_i \otimes d_i$ is a Hochschild cocycle. More precisely, it is the 2-cocycle on A associated to the cup products of the Hochschild 1-cocycles d_j, d_i on A; for more details see* [GM1, MW, MPSW].

We will now show that $\exp_q(\eta_{j,i})$ is a multiplicative cocycle (the q-exponential is defined with respect to the convolution product in $(A \otimes A)^* = A^* \otimes A^*$).

Theorem 4.6. *If i, j are linkable roots, $q = \chi_i(g_j)$, and $\eta_{j,i} \colon A \otimes A \to k$ is as above, then $\sigma := exp_q(\eta_{j,i}) \colon A \otimes A \to k$ is a multiplicative cocycle.*

Proof. We need to prove that

$$(\varepsilon \otimes \sigma) * \sigma(\mathrm{id} \otimes \mathrm{m}) = (\sigma \otimes \varepsilon) * \sigma(\mathrm{m} \otimes \mathrm{id}).$$

First note that $\varepsilon \otimes \sigma = \exp_q(\varepsilon \otimes \eta_{j,i})$, $\sigma \otimes \varepsilon = \exp_q(\eta_{j,i} \otimes \varepsilon)$, $\sigma(\mathrm{id} \otimes \mathrm{m}) = \exp_q(\eta_{j,i}(\mathrm{id} \otimes \mathrm{m}))$, and $\sigma(\mathrm{m} \otimes \mathrm{id}) = \exp_q(\eta_{j,i}(\mathrm{m} \otimes \mathrm{id}))$. The q-exponentials here are defined in the convolution algebra $(A \otimes A)^*$. Observe that for $r \in \{i, j\}$ we have $d_r \mathrm{m} = \Delta_{A^*} d_r = d_r \otimes \varepsilon + \chi_r \otimes d_r$ and that $\chi_r \mathrm{m} = \Delta_{A^*}\chi_r = \chi_r \otimes \chi_r$. Hence we have $\varepsilon \otimes \eta_{j,i} = \varepsilon \otimes d_j * \chi_i \otimes d_i$, $\eta_{j,i}(\mathrm{id} \otimes \mathrm{m}) = d_j * \chi_i \otimes d_i \otimes \varepsilon + d_j * \chi_i \otimes \chi_i \otimes d_i$, $\eta_{j,i} \otimes \varepsilon = d_j * \chi_i \otimes d_i \otimes \varepsilon$, and $\eta_{j,i}(\mathrm{m} \otimes \mathrm{id}) = d_j * \chi_i \otimes \chi_i \otimes d_i + \varepsilon \otimes d_j * \chi_i \otimes d_i$. If $a = \varepsilon \otimes d_j * \chi_i \otimes d_i$, $b = d_j * \chi_i \otimes d_i \otimes \varepsilon$, $c = d_j * \chi_i \otimes \chi_i \otimes d_i$ then an easy calculation shows that $a * b = b * a$, $c * a = qa * c$, $c * b = qb * c$, and $a^i * c^{\ell - i} = 0 = b^i * c^{\ell - i}$ for $i = 0, \ldots, \ell$. By Lemma 4.4 it now follows that

$$\begin{aligned}
(\varepsilon \otimes \sigma) * \sigma(\mathrm{id} \otimes \mathrm{m}) &= \exp_q(a) * \exp_q(b + c) \\
&= \exp_q(a) * \exp_q(b) * \exp_q(c) \\
&= \exp_q(b) * \exp_q(a) * \exp_q(c) \\
&= \exp_q(b) * \exp_q(a + c) \\
&= (\sigma \otimes \varepsilon) * \sigma(\mathrm{m} \otimes \mathrm{id})
\end{aligned}$$

as required. □

Lemma 4.7. *If (i, j) and (s, t) are disjoint pairs of linkable roots, then $\eta_{j,i}$ and $\eta_{s,t}$ commute, and thus also $\exp_{\chi_i(g_j)}(\eta_{j,i})$ and $\exp_{\chi_s(g_t)}(\eta_{t,s})$ commute.*

Proof. It follows directly from the fact that

$$\chi_i(g_t)\chi_t(g_i)\chi_j(g_s)\chi_s(g_j) = 1.$$

□

Let $G_0 = \left\{ g_\beta^{N_\beta} \middle| \beta \in \Phi^+, \chi_\beta^{N_\beta} = \varepsilon, g_\beta^{N_\beta} \neq 1 \right\}$. Let $\mathcal{D}'$ be the datum obtained from $\mathcal{D}$ by replacing G by $G' = G/G_0$. It is well know [AS] that elements of G_0 are central in $u(\mathcal{D}, \lambda, \mu)$ for all λ and μ. Abbreviate I_μ the ideal in $u(\mathcal{D}, 0, \mu)$ generated by $(kG_0)^+$ and note that $u(\mathcal{D}, 0, \mu)/I_\mu = u(\mathcal{D}', 0, 0)$. We will use π_μ to denote the canonical projection $\pi_\mu \colon u(\mathcal{D}, 0, \mu) \to u(\mathcal{D}, 0, \mu)/I_\mu = u(\mathcal{D}', 0, 0)$.

Proposition 4.8. *Let λ be a linking datum for $\mathcal{D}$ (and hence also for $\mathcal{D}'$). Let $\sigma_\lambda \colon u(\mathcal{D}', 0, 0) \otimes u(\mathcal{D}', 0, 0) \to k$ be the multiplicative cocycle from Lemma 4.7. Then $\sigma_{\lambda,\mu} := \sigma_\lambda \circ (\pi_\mu \otimes \pi_\mu) \colon u(\mathcal{D}, 0, \mu) \otimes u(\mathcal{D}, 0, \mu) \to k$ is a multiplicative cocycle and $u(\mathcal{D}, 0, \mu)_{\sigma_{\lambda,\mu}} = u(\mathcal{D}, \lambda, \mu)$.*

Proof. Proof is almost identical to the proof of Proposition 4.2. Let $B = u(\mathcal{D}, 0, \mu)_{\sigma_{\lambda,\mu}}$. Note that for $1 \leq i < j \leq \theta$ we have $m_B(x_i \otimes x_j - \chi_j(g_i) x_j \otimes x_i) = \lambda_{i,j}(1 - g_i g_j)$ and $m_B(x_i^{\otimes N_i}) = \mu_i(1 - g_i^{N_i})$. Hence $B = u(\mathcal{D}, 0, \mu)_{\sigma_{\lambda,\mu}}$ is a quotient of $A = u(\mathcal{D}, \lambda, \mu)$. Since the dimensions of A and B are the same we conclude that $u(\mathcal{D}, 0, \mu)_{\sigma_{\lambda,\mu}} = u(\mathcal{D}, \lambda, \mu)$. □

We can now explicitly describe all deforming cocycles for the quantum linear space.

Theorem 4.9. *Let $\sigma := \sigma_{\lambda,\mu} * e^{\sum_\beta \mu_i \zeta_i}$. Then σ is a multiplicative cocycle and $u(\mathcal{D}, 0, 0)_\sigma = u(\mathcal{D}, \lambda, \mu)$.*

Proof. Follows from Propositions 4.8 and 4.2. □

4.6. **Remarks regarding the general case.** The q-exponential map idea described above can be adjusted [GM2] to the general case (liftings of bosonizations of Nichols algebras of finite Cartan type) and one can thus obtain the explicit description of the cocycles deforming $u(\mathcal{D}, 0, \mu)$ to $u(\mathcal{D}, \lambda, \mu)$. In other words Proposition 4.8 extends to liftings of bosonizations of all Nichols algebras of finite Cartan type. The procedure is almost the same as the procedure for a quantum linear space. The only subtle point is to a carefully order the linkable roots in such a way that all q-exponentials involved commute (replacing $\exp_q(d_i \chi_j \otimes d_j)$ by $\exp_q(d_j \chi_i \otimes d_i)$ when necessary) . The same idea also works to explicitly describe the cocycles that deform the 'big' quantum groups, that is deforming $U(\mathcal{D}, 0)$ to $U(\mathcal{D}, \lambda)$.

Recently we have also been able to adapt our exponential map ideas to explicitly describe cocycles deforming $u(\mathcal{D}, 0, 0)$ to $u(\mathcal{D}, 0, \mu)$ for a much larger class of Andruskiewitsch-Schneider Hopf algebras. We are preparing this work as a separate publication [GM2]. There we also explain how all of the root vector cocycles can be obtained from Singer cocycles. We briefly describe this below: Let G_0 be the subgroup of G generated by those $g_i^{N_i}$ for which $\mu_i \neq 0$ and let $A = kG_0$. Note that A is a central Hopf subalgebra of $B = u(\mathcal{D}, 0, \mu)$ and that the 'quotient' is $C = u(\mathcal{D}', 0, 0)$. We thus get a Singer extension (also called cleft extension) [Si] $A \xrightarrow{i} B \xrightarrow{\pi} C$. Let $a \colon C \otimes C \to A$ be the algebra part of the associated Singer cocycle (it can be computed explicitly; see below). Then for any character $\phi \colon G_0 \to k$

we get a multiplicative cocycle $\sigma' = \phi \circ a \colon C \otimes C \to k$. This cocycle naturally lifts from $C = u(\mathcal{D}', 0, 0)$ to $u(\mathcal{D}, 0, 0)$ and the cocycle twist with the above multiplicative cocycle yields $u(\mathcal{D}, 0, \mu')$, where the datum μ' is the appropriately re-scaled datum μ (depending on the choice of the character ϕ).

How to get a explicitly [An, By]: Let $\xi \colon B \to A$ denote the section of i that sends PBW-elements from G_0 to themselves and is ε on all other PBW-elements. As ξ is a unit and counit preserving A-module map we can define a C-comodule map $\chi \colon C \to B$ by $\chi\pi = (i \circ \xi^{-1}) * id$. Map $a \colon C \otimes C \to A$ is then given by $a(x, y) = \chi(x_1)\chi(y_1)\chi^{-1}(x_2 y_2)$ (the range falls into the C-coinvariant part of B, which is $i(A)$).

Combining the linking cocycles obtained by the q-exponential formula with the root-vector cocycles obtained from Singer cocycles then describes explicitly all deforming cocycles for all of the Andruskiewitsch-Schneider Hopf algebras.

5. Examples

The following examples of explicit description of deforming cocycles of liftings of various quantum linear spaces illustrate the ideas and discussion above. The first (and simplest) example is worked out in a lot of detail, implicitly repeating many of the more general constructions above. This was originally done by the authors in order to better understand the lifting process and gain insight into the structure of liftings. We hope that the reader will similarly benefit from the experience.

Example 5.1. Let $G = \langle g \rangle$ be a cyclic group of order Np and let $V = kx$ be a 1-dimensional crossed G-module with action and coaction given by $gx = qx$ for a primitive N-th root of unity q and $\delta(x) = g \otimes x$. The braiding $c \colon V \otimes V \to V \otimes V$ is then determined by $c(x \otimes x) = qx \otimes x$. The braided Hopf algebra $\mathcal{A}(V)$ is the polynomial algebra $k[x]$ with comultiplication $\Delta(x^i) = \sum_{r+s=i} \binom{i}{r}_q x^r \otimes x^s$ in which x^N is primitive. The braided Hopf algebra $\mathcal{C}(V) = k\langle x \rangle$ is the divided power Hopf algebra with basis $\{ x_i | \, i \geq 0\}$, comultiplication $\Delta(x_i) = \sum_{r+s=i} x_r \otimes x_s$ and multiplication $x_i x_j = \binom{i+j}{i}_q x_{i+j}$. The quantum symmetrizer $\mathcal{S} \colon \mathcal{A}(V) \to \mathcal{C}(V)$ is given by $\mathcal{S}(x^i) = \mathcal{S}(x)^i = i_q! x_i$. The Nichols algebra of V is $B(V) = \mathcal{A}(V)/(x^N) \cong \operatorname{im} \mathcal{S}$ and the Hopf algebra

$$\begin{aligned} A &= B(V)\#kG \\ &= \left\langle x, g | x^N = 0, g^{Np} = 1, gx = qxg, \Delta(x) = x \otimes 1 + g \otimes x, \Delta(g) = g \otimes g \right\rangle \end{aligned}$$

is coradically graded.

The linear functional $\zeta \colon A \otimes A \to k$ of degree $-N$, defined by

$$\zeta(x^i g^j \otimes x^k g^l) = a q^{jk} \delta_N^{i+k},$$

is a Hochschild cocycle with $\zeta^2 = 0$, and satisfying

$$\zeta(m \otimes 1) * (\zeta \otimes \varepsilon) = \zeta(1 \otimes m) * (\varepsilon \otimes \zeta).$$

It follows that

$$\sigma = e^{\zeta} = \varepsilon \otimes \varepsilon + \zeta \colon A \otimes A \to k$$

is a convolution invertible multiplicative cocycle. In terms of the dual basis of A^* it can be expressed as

$$\sigma = \varepsilon \otimes \varepsilon + \zeta = \varepsilon \otimes \varepsilon + \sum_{r+s=N}^{0<r,s<N} a_{rs} \xi^r \theta^s \otimes \xi^s,$$

where $a_{rs} = \frac{1}{r_q! s_q!}$. The corresponding cocycle deformation of the multiplication of A is

$$m_\sigma = (\sigma \otimes m \otimes \sigma^{-1})\Delta^{(2)}_{A\otimes A} = m + (\zeta \otimes m - m \otimes \zeta)\Delta_{A\otimes A},$$

since $(\zeta \otimes m \otimes \zeta)\delta_{A\otimes A} = 0$ (it is of degree $-2N$). Using

$$\Delta_{A\otimes A}(x^i g^j \otimes x^k g^l) = \sum_{r+s=i}^{u+v=k} \binom{i}{r}_q \binom{k}{u}_q x^r g^{s+j} \otimes x^u g^{v+l} \otimes x^s g^j \otimes x^v g^l$$

and invoking the identity [Ka]

$$\sum_{s+v=\beta} \binom{i}{s}_q \binom{k}{v}_q q^{s(k-v)} = \binom{i+k}{\beta}_q = \binom{N\alpha+\beta}{\beta}_q = 1$$

when $i + k = N\alpha + \beta$, the following explicit formula for m_σ can be deduced:

$$\begin{aligned}
m_\sigma(x^i g^j \otimes x^k g^l) &= q^{jk} x^{i+k} g^{j+l} \\
&+ \sum_{r+s=i}^{u+v=k} a \binom{i}{r}_q \binom{k}{u}_q q^{(s+j)u+jv} (\delta^{r+u}_{n\alpha} x^{s+v} - x^{r+u} \delta^{s+v}_{n\alpha} g^{s+v}) g^{j+l} \\
&= q^{jk} \Bigg[x^{i+k} + a x^\beta \Big(\sum_{s+v=\beta} \binom{i}{s}_q \binom{k}{v}_q q^{s(k-v)} \\
&\quad - \sum_{r+u=\beta} \binom{i}{r}_q \binom{k}{u}_q q^{(i-r)u} g^{N\alpha} \Bigg] g^{j+l} \\
&= q^{jk} \left[x^{i+k} + a x^\beta (1 - g^{N\alpha}) \right] g^{j+l},
\end{aligned}$$

where $i + k = N\alpha + \beta$ with $\alpha = 0, 1$.

The linear dual $V^* = k\xi$, $\xi(x) = 1$, is a crossed module over the character group $\widehat{G} = \langle \theta \rangle$, $\theta(g) = \alpha$ a primitive Np-th root of unity and $\alpha^p = q$, with action $\delta^*(\theta \otimes \xi) = \theta\xi = \alpha\xi$ and coaction $\mu^*(\xi) = \phi \otimes \xi$, where $\phi = \theta^p$. The graded braided Hopf algebra $\mathcal{A}(V^*) \cong k[\xi] \cong \mathcal{C}(V)^*$ is the graded polynomial algebra with comultiplication $\Delta(\xi^i) = \sum_{r+s=i} \binom{i}{r}_q \xi^r \otimes \xi^s$ so that ξ^N is primitive. The cofree graded braided Hopf algebra $\mathcal{C}(V^*) = k\langle \xi \rangle \cong \mathcal{A}(V)^*$ is the divided power Hopf algebra with basis $\{\xi_i | \, i \geq 0\}$, comultiplication $\Delta(\xi_i) = \sum_{r+s=i} \xi_r \otimes \xi_s$ and multiplication $\xi_i \xi_j = \binom{i+j}{i}_q \xi_{i+j}$. The quantum symmetrizer $\mathcal{S} \colon \mathcal{A}(V^*) \to \mathcal{C}(V^*)$ is

given by $\mathcal{S}(\xi^i) = i_q!\xi_i$. The Nichols algebra of V^* is $B(V^*) = \mathcal{A}(V^*)/(\xi^N) \cong \operatorname{im}\mathcal{S}$ and the Hopf algebra

$$\begin{aligned} A^* &= B(V^*)\#k\widehat{G} \\ &= \left\langle \xi,\theta \,|\, \xi^N = 0, \theta^{Np} = \varepsilon, \theta\xi = \alpha\xi\theta, \Delta(\xi) = \xi\otimes\varepsilon + \phi\otimes\xi, \Delta(\theta) = \theta\otimes\theta \right\rangle \end{aligned}$$

is radically graded.

The invertible element $\sigma^*\colon k \to A^*\otimes A^*$ with $\sigma^*(1) = \sigma = \varepsilon\otimes\varepsilon + \sum_{r+s=n} a_{rs}\xi^r\phi^s\otimes\xi^s = \varepsilon\otimes\varepsilon + \zeta$, with $a_{rs} = \frac{1}{r_q!s_q!}$, is the cocycle above represented in terms of the basis of A^*. Observe that ζ is of degree n and $\zeta^2 = 0$. The resulting cocycle deformation of the comultiplication

$$\Delta_\sigma = m^{(2)}_{A\otimes A}\left(\sigma\otimes\Delta\otimes\sigma^{-1}\right) = \Delta + m_{A\otimes A}(\zeta\otimes\Delta - \Delta\otimes\zeta) = \delta_0 + \delta_n,$$

where $m^{(2)}_{H\otimes H}(\zeta\otimes\Delta\otimes\zeta) = 0$ is used, is compatible with the original multiplication. Since $\Delta(\theta)\zeta = \alpha^N\zeta\Delta(\theta)$, it follows that

$$\Delta_\sigma(\theta^i) = \theta^i\otimes\theta^i + (1-\alpha^{Ni})\zeta(\theta^i\otimes\theta^i).$$

Using the identity $a_{u-1,s}q^s + a_{u,s-1} = a_{u,s}$ one finds that $\zeta\Delta(\xi) = \Delta(\xi)\zeta$, so that $\Delta_\sigma(\xi) = \Delta(\xi)$ and

$$\Delta_\sigma(\xi^i\theta^j) = \Delta(\xi^i)\Delta_\sigma(\theta^j).$$

The deformed Hopf algebra has the presentation

$$A^\sigma = \left\langle \xi,\theta \,|\, \Delta_\sigma(\xi) = \Delta(\xi), \Delta_\sigma(\theta) = \theta\otimes\theta + (1-\alpha^N)\zeta(\theta\otimes\theta) \right\rangle$$

with the original multiplication and radical filtration, so that $\operatorname{gr}_r A^\sigma = A^*$.

Example 5.2 (cf. [ABM])**.** This is a 2-variable analogue of the example above. Let $G = \langle g\rangle$ be the cyclic group of order mn and q a primitive m-th root of unity. Consider the 2-dimensional crossed G module $V = kx_1\oplus kx_2$ with action $gx_i = q^{(-1)^{i-1}}x_i$ and coaction $\delta(x_i) = g\otimes x_i$. The braiding map $c\colon V\otimes V\to V\otimes V$ is then $c(x_i\otimes x_j) = q^{(-1)^{j-1}}x_j\otimes x_i$.

The bosonization of the Nichols algebra is then

$$A = B(V)\#kG = \langle g, x_1, x_2 \,|\, g^{mn} = 1, x_1^m = 0, x_2^m = 0, x_2x_1 = qx_1x_2\rangle$$

with comultiplication $\Delta(g) = g\otimes g$ and $\Delta(x_i) = x_i\otimes 1 + g\otimes x_i$. Note that it is coradically graded.

The linear functionals $\zeta_i\colon A\otimes A\to k$ of degree $-m$, given by

$$\zeta_i(x_1^{j_1}x_2^{j_2}g^r\otimes x_1^{j_1'}x_2^{j_2'}g^{r'}) = \delta_{j_i+j_i',m}\delta_{j_{i+1}+j_{i+1}',0}q^{(-1)^i rj_i'}$$

are commuting Hochschild cocycles and the cocycle deformation associated to the multiplicative cocycle $\sigma_{\mu_1,\mu_2} = e^{\mu_1\zeta_1+\mu_2\zeta_2} = \varepsilon + \mu_1\zeta_1 + \mu_2\zeta_2 + \mu_1\mu_2\zeta_1\zeta_2$ has presentation

$$A_{\sigma_{\mu_1,\mu_2}} = \langle g, x_1, x_2 \,|\, g^{mn} = 1, x_1^m = \mu_1(1-g^m), x_2^m = \mu_2(1-g^m), x_2x_1 = qx_1x_2\rangle\,.$$

Moreover, if $d_i\colon A \to k$ are skew-derivations given by

$$d_1(x_1^{a_1}x_2^{a_2}g) = \begin{cases} 1; a_1 = 1, a_2 = 0 \\ 0;\ \text{otherwise} \end{cases} ,\ d_2(x_1^{a_1}x_2^{a_2}g) = \begin{cases} 1; a_1 = 0, a_2 = 1 \\ 0;\ \text{otherwise} \end{cases}$$

then $\sigma = \sigma_{\lambda;\mu_1,\mu_2} = \exp_q(\lambda d_2\chi \otimes d_1) * \sigma_{\mu_1,\mu_2}$ is a multiplicative cocycle and A_σ has presentation

$$\left\langle g, x_1, x_2 |\, g^{mn} = 1, x_1^m = \mu_1(1-g^m), x_2^m = \mu_2(1-g^m), x_2x_1 - qx_1x_2 = \lambda(1-g^2) \right\rangle.$$

References

[An] N. Andruskiewitsch, *Notes on extensions of Hopf algbebras*, Can. J. Math. Vol 48 (1) (1996), 3–42.

[ABM] A. Ardizzoni, M. Beattie, C. Menini, *Cocycle deformations for Hopf algebras with a coalgebra projection*, preprint, `arXiv:math.QA/0906.0239 v2`, to appear in J. Algebra,

[AS] N. Andruskiewitsch and H.J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, preprint, `arXiv:math.QA/0502157 v3`, 2006, Ann. Math **171** (2010), No 1, 375-417.

[AS1] N. Andruskiewitsch and H.J. Schneider, *Finite Quantum groups and Cartan matrices*, Adv. in Math. **154** (2000), 1–45.

[AS2] N. Andruskiewitsch and H.J. Schneider, *Finite Quantum groups over abelian groups of prime exponent*, Ann. Sci. Ec. Norm. Sup. **35** (2002), 1–26.

[AS3] N. Andruskiewitsch and H.J. Schneider, *Pointed Hopf Algebras*, in: Recent developments in Hopf algebra Theory, MSRI Publications **43** (2002), 1–68, Cambridge Univ. Press.

[BDG] M. Beattie, S. Dăscălescu, L. Grunenfelder, *On the number of types of finite-dimensional Hopf algebras*, Invent. Math. **136** (1999), no. 1, 1–7.

[BDR] M. Beattie, S. Dascalescu, S. Raianu, *Liftings of Nichols algebras of type B_2*, Israel J. Math. **132** (2002), 1–28.

[By] N. P. Byott, *Cleft extensions of Hopf algebras*, J. Algebra **157** (1993), 405–429.

[CP] V. Chari and A. Pressley, (A Guide to) Quantum groups, Cambridge University Press, 1994.

[Di] D. Didt, *Pointed Hopf algebras and quasi-isomorphisms*, Algebra and Rep. Theory **8** (2005), 347–362.

[DCY] Y. Du, X.-W. Chen, and Y. Ye, *On graded bialgebra deformations*, Algebra Colloq. 14 (2007), no. 2, 301312.

[EG] P. Etingof and S. Gelaki, *On families of triangular Hopf algebras*, Int. Math. Res. not. **2002**, 757–768.

[Gra] M. Graña, *Pointed Hopf algebras of dimension 32.* Comm. Algebra **28** (2000), 2935-2976.

[Gr] L. Grunenfelder, *About Braided and Ordinary Hopf Algebras*, Milan j. math. **71** (2003), 1–19.

[Gr2] L. Grunenfelder, *Cohomological aspects of Hopf algebra liftings*, preprint, `arXiv:math.RA/0909.4112v1`

[GS] M. Gerstenhaber and S.D. Shack, *Bialgebra cohomology, deformations, quantum groups and algebraic deformations*, Proc. Natl. Acad. Sci. USA, vol. 87 (1990), 478–481.

[GM1] L. Grunenfelder and M. Mastnak, *Pointed and copointed Hopf algebras as cocycle deformations*, `arXiv:math.RA/0709.0120`, 2007.

[GM2] L. Grunenfelder, M. Mastnak, *Exponential map, its Q-analogue, and cocycles*, in preparation.

[Ka] C. Kassel, Quantum groups, Graduate Texts in Mathematics, 155, Springer-Verlag, New York, 1995.

[Ma] A. Masuoka, *Defending the negated Kaplansky conjecture*, Proc. Amer. Math. Soc **129** (2001), 3185–3192.

[Ma1] A. Masuoka, *Abelian and non-abelian second cohomology of quantized enveloping algebras*, J. Algebra 320 (2008), no. 1, 1–47.

[MPSW] , M. Mastnak, J. Pevtsova, P. Schauenburg, S. Witherspoon, *Cohomology of finite-dimensional pointed Hopf algebras*, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 377–404.

[Mo] S. Montgomery, Hopf algebras and their actions on rings, CBMS Conf. Series in Math., **82**, AMS, Providence, RI, 1993.

[MW] M. Mastnak and S. Witherspoon, *Bialgebra cohomology, pointed Hopf algebras and deformations*, J. Pure Appl. Algebra 213 (2009), no. 7, 1399–1417.

[Mu] E. Mueller, *The Coradical filtration of $U_q(\mathfrak{g})$ at roots of unity*, Com. in Algebra **28** (2000), 1020–1044.

[Ra] D. Radford, *Hopf algebras with a projection*, J. Algebra **1985**, 322–347.

[Ri] C. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–591.

[Sch] P. Schauenburg, *Hopf-Galois extensions*, Comm. Algebra **24** (1996), 3797–3825.

[Si] W. M. Singer, *Extension theory for connected Hopf algebras*, J. Algebra **21**(1972), 1–16.

[W] S. Witherspoon, *Skew derivations and deformations of a family of group crossed products*, Comm. Algebra 34 (2006), no. 11, 4187-4206.

Department of Mathematics, The University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: luzius@math.ubc.ca, luzius@mathstat.dal.ca

Department of Mathematics and C.S., Saint Mary's University, Halifax, NS B3H 3C3, Canada

E-mail address: mmastnak@cs.smu.ca